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A lattice formulation of super Yang-Mills theories with exact supersymmetry

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Abstract: We construct super Yang-Mills theories with extended supersymmetry on hypercubic lattices of various dimensions keeping one or two supercharges exactly. Gauge fields are represented by ordinary unitary link variables, and the exact supercharges are nilpotent up to gauge transformations. Among the models, we show that the desired continuum theories are obtained without any fine tuning of parameters for the cases $\mathcal{N} = 2, 4, 8$ in two-dimensions.

Keywords: Topological Field Theories, Lattice Quantum Field Theory, Lattice Gauge Field Theories, Extended Supersymmetry.

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1. Introduction

Nonperturbative aspects in supersymmetric gauge theory have been vigorously investigated, in particular since finding the exact form of the low energy effective action in four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory [1] and the AdS/CFT duality between gauge theory and gravity (string theory) [2]. For some operators such as the BPS saturated objects or the chiral operators, which are protected from the quantum corrections by certain algebraic properties, we can know exact results for their spectra or the correlation functions. The approach to the nonperturbative physics is based on the property of duality.

A conventional but more universal approach is lattice formulation, which enables numerical analysis for any observables not restricted to special operators with the BPS saturated or chiral properties. However, there has been difficulty on the lattice approach to supersymmetry, because of lack of infinitesimal translational invariance on the lattice and breakdown of the Leibniz rule [3, 4]. In spite of the difficulty, it is possible to construct lattice models, which do not have manifest supersymmetry but flow to the desired supersymmetric theories in the continuum limit. One of the examples is $\mathcal{N}=1$ super Yang-Mills (SYM) theory in four-dimensions whose field contents are gauge bosons and gauginos. Since in the theory the only relevant supersymmetry breaking operator is the gaugino mass, one can arrive at the supersymmetric continuum theory if the radiative corrections are not allowed to induce the relevant operator by symmetries realized in the lattice theory. Making use of domain wall or overlap fermions keeps discrete chiral symmetry on lattice, which is the symmetry excluding the fermion mass [5].

Supersymmetric theories with extended supersymmetry have some supercharges, which are not related to the infinitesimal translations and can be seen as fermionic internal symmetries. It is possible to realize a part of such supercharges as exact symmetry on lattice, and the exact supersymmetry plays a key role to restore the full supersymmetry in the continuum limit without any fine tuning of parameters. One of the examples is $\mathcal{N}=2$ Wess-Zumino model in two-dimensions where the Nicolai mapping was heavily utilized to construct the lattice model [6]. Also for SYM theories with extended supersymmetry, Cohen, Kaplan, Katz and Ünsal recently proposed such a kind of various lattice models motivated by the idea of deconstruction [7, 8].[1] In these models, to generate the kinetic terms of the target theories and to stabilize noncompact bosonic zero-modes (the so-called radions), one has to add terms softly breaking the exact supersymmetry, which are tuned to vanish in large volume limit.

In many cases, the above 'internal' supersymmetries can be reinterpreted as the BRST symmetries in topological field theories. So, in order to construct lattice models respecting the 'internal' supersymmetry, it seems natural to start with the topological field theory formulation of the theory. In ref. [11], Catterall discussed on a general formulation of lattice models based on the connection to topological field theory for supersymmetric theories without gauge symmetry. Two-dimensional $\mathcal{N}=2$ Wess-Zumino model was also investigated there.

Here, standing on the same philosophy, we construct lattice models for SYM theories with extended supersymmetry keeping one or two supercharges exactly. Our models are motivated by the topological field theory formulation of $\mathcal{N}=2,4$ SYM theories, and free from the radion problems. The lattices have hypercubic structures, and the gauge fields are expressed as ordinary compact unitary variables on the lattice links. Our lattice models have a huge degeneracy of the classical vacua growing as exponential of the number of plaquettes, which causes difficulty for perturbative analysis around a specific vacuum. From the reason not completely same but somewhat similar to the case [7, 8], we add terms

[1] For some related works, see refs. [9]. Also, for other attempts to lattice formulations of supersymmetry, see [10, 11].

breaking the exact supersymmetry to pick up a single vacuum corresponding to the desired continuum theory.

The classical actions of $\mathcal{N} = 2$ SYM theories are written as the 'topological field theory (TFT) form': Q(something). Q is one of the supercharges of the SYM theories and nilpotent up to gauge transformations. Topological field theories are related to the $\mathcal{N} = 2$ theories via the procedure of twist [12]. The twist changes the spin quantum numbers of fields, and some of supercharges become scalar. When regarding one of the scalar supercharges Q as the BRST charge whose cohomology defines physical observables, the theories become topological field theories. In this paper, however we do not change the interpretation of the supercharge Q. Namely, Q is regarded as just one of spinor supercharges of the original SYM theories not as the BRST operators. For theories on the flat space, the twisted theories are equivalent to the original theories, and the twist is solely renaming the field variables. We construct the lattice theories for $\mathcal{N} = 2$ SYM theories in the dimensions $d = 2,\ 4,\ 8$ keeping the exact supercharge Q. The lattice actions have the Q-exact form as well as the continuum ones. Among them, the $d = 2$ case is shown to flow to the desired continuum theory without any fine tuning of parameters, where the supersymmetry breaking term does not alter the result of renormalization.

For $\mathcal{N} = 4$ SYM theories, the actions are written as the form: Q_+Q_-(something), where $Q_\pm$ are two of the supercharges and nilpotent up to gauge transformations. We refer it as 'balanced topological field theory (BTFT) form'. Balanced topological field theories are also related to $\mathcal{N} = 4$ SYM theories via the twist [13, 14, 15]. Similarly to the $\mathcal{N} = 2$ case, we regard $Q_\pm$ as two of spinor supercharges of the $\mathcal{N} = 4$ theories, and the twist amounts just to changing the notations of fields. The lattice models are constructed for $\mathcal{N} = 4$ SYM in two- and four-dimensions preserving the $Q_\pm$ supersymmetries exactly. In this case, the lattice actions have $\mathrm{SU}(2)_R$ symmetry transforming $Q_\pm$ as a doublet, which is a subgroup of SU(4) internal symmetry of $\mathcal{N} = 4$ theories. We show that the lattice theory for $\mathcal{N} = 4$ in two-dimensions defines the corresponding continuum theory with no requirement for fine tuning as well as $\mathcal{N} = 8$ theory in two-dimensions which is obtained by dimensional reduction from the $\mathcal{N} = 4$ in four-dimensions.

This paper is organized as follows. In section 2, we briefly explain on the 'topological field theory form' of $\mathcal{N} = 2$ SYM theories. In section 3, we construct the lattice models for $\mathcal{N} = 2$ theories in $d = 2, 4, 8$, and show that the continuum theory is defined without fine tuning in the case $d = 2$. In section 4, we give some explanations on the 'balanced topological field theory form' for $\mathcal{N} = 4$ SYM theories in two- and four-dimensions. In section 5, we formulate the $\mathcal{N} = 4$ theories on hypercubic lattices, and discuss on the renormalization in the continuum limit. Section 6 is devoted to summary and discussions. In appendix A, we give the explicit form of the γ-matrix conventions used in the paper. In appendix B, we present the explicit form of the matrices P_μ for the $\mathcal{N} = 2$ and 4 theories, which appear in the term analogous to the Wilson term removing the species doublers of fermions. In appendix C, we discuss about $Q_\pm$-cohomology in the $\mathcal{N} = 4$ lattice models, whose result is useful for the renormalization argument in section 5.

Throughout this paper, we focus on the gauge group $G = \mathrm{SU}(N)$. In sections 2 and 4 discussing continuum theories, notations of repeated indices in formulas are assumed to be summed. On the other hand, in sections 3 and 5 for lattice theories, we explicitly write the summation over the indices except the cases of no possible confusion.

2. 'Topological field theory form' of $\mathcal{N}=2$ super Yang-Mills theories

First, we give some explanation on Wick-rotated $\mathcal{N}=2$ SYM theories [16], and then move on their description in the 'topological field theory form'.

2.1 Note on Wick-rotation and $\mathrm{U}(1)_R$ symmetry

We start with $\mathcal{N}=1$ SYM in $(D-1,1)$-dimensions $(D=4,6,10)$:

$$S^{(M)}=\frac{1}{g^2}\int \mathrm{d}^D x\,\mathrm{tr}\left[-\frac{1}{4}F^{MN}F_{MN}-\frac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi\right], \tag{2.1}$$

where $M,N=0,1,\ldots,D-1$, and the metric tensor is $\eta^{MN}=\mathrm{diag}(-1,1,1,\ldots,1)$. The Γ-matrices satisfy the Clifford algebra $\{\Gamma^M,\Gamma^N\}=-2\eta^{MN}$. Here we take the Majorana representation for Γ^M. Namely, all of Γ^M consist of pure imaginary entries. Γ^0 is anti-symmetric, while Γ^i $(i=1,\ldots,D-1)$ are symmetric. The spinor field is expanded as $\Psi=\sum_a\Psi^a T^a$. Ψ^a are real Grassmann valued spinors, and T^a represent a basis of the gauge group generators. The number of spinor components is $2(D-2)$. Note

$$\bar{\Psi}=\Psi^T\Gamma^0\,, \tag{2.2}$$

where the transpose acts only on spinor indices. The action is invariant under the Lorentz boost along the x^i direction transforming the fermion as

$$\Psi\to e^{\Gamma^0\Gamma^i\frac{\theta}{2}}\Psi\,. \tag{2.3}$$

After the Wick-rotation ($x^0\to -ix_D$, $\Gamma_D\equiv i\Gamma^0$), the euclidean action reads

$$S^{(E)}=\frac{1}{g^2}\int \mathrm{d}^D x\,\mathrm{tr}\left[\frac{1}{4}F_{MN}F_{MN}+\frac{i}{2}\bar{\Psi}\Gamma_M D_M\Psi\right], \tag{2.4}$$

with $\{\Gamma_M,\Gamma_N\}=-2\delta_{MN}$ $(M,N=1,\ldots,D)$. Now the Lorentz boost (2.3) becomes a rotation in (i,D)-plane:

$$\Psi\to e^{\Gamma_D\Gamma_i\frac{\theta}{2}}\Psi\,. \tag{2.5}$$

This rotation does not keep the property of Ψ^a being real. Nevertheless, we can give a justification to regard the rotation as a symmetry of the action by still using (2.2) for $\bar{\Psi}$ in the action (2.4). We can interpret Ψ^a as complexified Grassmann and the action as holomorphic with respect to Ψ^a without changing the result of the path integral.[2] In the complexified space, (2.5) is closed and the rotation can be understood as a symmetry of the action.

The action (2.4) is invariant under the supersymmetry transformation:

$$\delta A_M=\epsilon^T\gamma_M\Psi,\quad \delta\Psi=-\frac{1}{2}(F_{ij}\gamma_{ij}+2F_{iD}\gamma_i)\epsilon\,, \tag{2.6}$$

where ϵ is a real spinor parameter. Also, $\gamma_i\equiv -i\Gamma^0\Gamma_i$, $\gamma_D\equiv\mathbf{1}$, $\gamma_{ij}=\frac{1}{2}[\gamma_i,\gamma_j]$. Then, γ_i satisfy $\{\gamma_i,\gamma_j\}=-2\delta_{ij}$.

[2] A complexified Grassmann number takes the form: (complex number) $\times$ (real Grassmann number).

As a result of dimensional reduction with respect to the x^{D-1} and x^D directions, the theory (2.4) becomes $\mathcal{N}=2$ SYM theory in $d \equiv (D-2)$-dimensions. The rotational symmetry in $(D-1,D)$-plane is seen as $\mathrm{U}(1)_R$ internal symmetry in the $\mathcal{N}=2$ theory. Also, the $\mathcal{N}=2$ supersymmetry is given by the dimensional reduction of (2.6).

2.2 'Topological field theory form'

The result of the dimensional reduction of the action (2.4) can be written as the 'topological field theory (TFT) form' [12, 17, 18]:

$$S_{\mathcal{N}=2} = Q\frac{1}{2g_d^2}\int \mathrm{d}^d x\,\mathrm{tr}\left[\frac{1}{4}\eta\,[\phi,\,\bar{\phi}] - i\vec{\chi}\cdot\vec{\Phi} + \vec{\chi}\cdot\vec{H} - i\psi_\mu D_\mu\bar{\phi}\right], \tag{2.7}$$

where μ is the index for d-dimensional space-time. Bosonic fields are A_μ, $\phi \equiv A_{D-1}+iA_D$, $\bar{\phi} \equiv A_{D-1}-iA_D$, and the auxiliary fields $\vec{H}$. The other fields ψ_μ, $\vec{\chi}$, η are fermionic. $\vec{H}$ and $\vec{\chi}$ are column vectors with $(d-1)$-components. $\vec{\Phi}$ are functions of field strengths $F_{\mu\nu}$ and have one, three, seven components for the cases $d=2,4,8$, respectively. Their forms are given by

$$\Phi \equiv 2F_{12} \qquad \text{for } d=2 \tag{2.8}$$

$$\Phi_{\mathtt{A}} \equiv 2\left(F_{\mathtt{A}4} + \frac{1}{2}\varepsilon_{\mathtt{ABC}}\,F_{\mathtt{BC}}\right) \quad \text{for } d=4 \tag{2.9}$$

$$\Phi_{\mathtt{A}} \equiv 2\left(F_{\mathtt{A}8} + \frac{1}{2}c_{\mathtt{ABC}}\,F_{\mathtt{BC}}\right) \quad \text{for } d=8\,. \tag{2.10}$$

$c_{\mathtt{ABC}}$ are totally antisymmetric and expressed as the structure constants of the algebra of octonions. Nonzero components are

$$c_{127} = c_{163} = c_{154} = c_{253} = c_{246} = c_{347} = c_{567} = 1 \tag{2.11}$$

and those obtained by index permutations therefrom.

Q is one of the supercharges for the transformation (2.6), and its transformation rule is given as

$$\begin{aligned} QA_\mu &= \psi_\mu\,, & Q\psi_\mu &= iD_\mu\phi\,,\\ Q\phi &= 0\,, & & \\ Q\vec{\chi} &= \vec{H}\,, & Q\vec{H} &= [\phi,\,\vec{\chi}]\,,\\ Q\bar{\phi} &= \eta\,, & Q\eta &= [\phi,\,\bar{\phi}]\,. \end{aligned} \tag{2.12}$$

Q is nilpotent up to infinitesimal gauge transformations with the parameter ϕ. By choosing the representation of γ_i so that Ψ takes the form

$$\Psi^T = \left(\psi_1,\ldots,\psi_d,\chi_1,\ldots,\chi_{d-1},\frac{1}{2}\eta\right), \tag{2.13}$$

Q is expressed as the transformation (2.6) with $\epsilon^T = (0,\ldots,0,-\varepsilon)$. ε is a real Grassmann number. See appendix A for the explicit form of γ_i. Since in the path integral we may regard ϕ and $\bar{\phi}$ as independent variables, it is not necessary to worry about the asymmetry with respect to ϕ and $\bar{\phi}$ in the transformation (2.12).

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The action (2.7) is written more explicitly as

$$S_{\mathcal{N}=2} = \frac{1}{2g_d^2}\int \mathrm{d}^d x \,\mathrm{tr}\left[\frac{1}{4}[\phi,\,\bar{\phi}]^2 + \vec{H}\cdot\vec{H} - i\vec{H}\cdot\vec{\Phi} + D_\mu\phi D_\mu\bar{\phi} - \frac{1}{4}\eta\,[\phi,\,\eta] - \right.$$
$$\left. - \vec{\chi}\cdot[\phi,\,\vec{\chi}] + \psi_\mu[\bar{\phi},\psi_\mu] + i\vec{\chi}\cdot Q\vec{\Phi} + i\psi_\mu D_\mu\eta\right]. \qquad (2.14)$$

The charge for the $\mathrm{U}(1)_R$ rotation is assigned as follows: $+2$ for ϕ, -2 for $\bar{\phi}$, $+1$ for ψ_μ, -1 for $\vec{\chi}$ and η, 0 for A_μ and $\vec{H}$. As is seen from the Yukawa interactions in (2.14), the $\mathrm{U}(1)_R$ is a chiral rotation which suffers anomaly generically in cases[3] $d = 4, 8$.

In refs. [12], topological field theory was obtained via the procedure of twist from theory with $\mathcal{N} = 2$ supersymmetry. The twist changes the spin of fields and some supercharges become scalar. If one of the scalar supercharges is regarded as the BRST charge, whose cohomology and the gauge invariance determine physical observables, the theory becomes topological field theory. In the context of topological field theory, the $\mathrm{U}(1)_R$ charge stands for the ghost number, and the $\mathrm{U}(1)_R$ anomaly is seen as the formal dimension of instanton moduli space. The instanton is defined by configurations satisfying the equations $\vec{\Phi} = 0$. Through the index theorem, the difference of the number of ψ_μ zero-modes (having the $\mathrm{U}(1)_R$ charge $+1$) and the number of $\vec{\chi}$, η zero-modes (having the charge -1) gives the formal dimension of the instanton moduli space.

Note that in this paper we regard the theory (2.7) as $\mathcal{N} = 2$ SYM theory itself, not as topological field theory. In fact, (2.7) is nothing but the $\mathcal{N} = 2$ SYM action after renaming the fermionic fields as (2.13)[4] up to total derivative terms. Here, we consider Q as one of the spinor supercharges and do not use it to define physical observables. Physical operators are determined solely by gauge invariance of the $\mathcal{N} = 2$ SYM theory. From this view-point, the topological field theory is considered as a special subsector of the original $\mathcal{N} = 2$ supersymmetric theory.

The total derivative terms exist in the 'TFT form' (2.7) for the cases $d = 4, 8$. They are explicitly written as

$$\begin{aligned} &\frac{1}{2g_4^2}\int \mathrm{tr}(F\wedge F) \quad \text{for } d = 4\,, \\ &\frac{1}{2g_8^2}\int \Omega\wedge\mathrm{tr}(F\wedge F) \quad \text{for } d = 8\,, \end{aligned} \qquad (2.15)$$

where $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, and Ω is a self-dual closed four-form:

$$\begin{aligned} \Omega &= \omega + *\omega \\ \omega &= \frac{1}{3!}\,c_{\mathtt{ABC}}\,\mathrm{d}x^{\mathtt{A}}\wedge\mathrm{d}x^{\mathtt{B}}\wedge\mathrm{d}x^{\mathtt{C}}\wedge\mathrm{d}x^8\,. \end{aligned} \qquad (2.16)$$

[3]In two-dimensions, since the Pontryagin class is tr F, it vanishes for $G = \mathrm{SU}(N)$ and the chiral symmetry $\mathrm{U}(1)_R$ is not anomalous.

[4]Note that the index μ of ψ_μ does not reflect the Lorentz transformation property.

3. Lattice formulation with one exact supercharge

In this section, we formulate the theory (2.7) on the d-dimensional hypercubic lattice keeping the supersymmetry Q. In the lattice theory, gauge fields $A_\mu(x)$ are promoted to the compact unitary variables

$$U_\mu(x) = e^{iaA_\mu(x)} \tag{3.1}$$

on the link $(x, x+\hat{\mu})$. 'a' stands for the lattice spacing, and $x \in \mathbf{Z}^d$ the lattice site. All other variables are distributed at sites. Interestingly, the Q-transformation (2.12) is extendible to the lattice variables preserving the property

$$Q^2 = (\text{infinitesimal gauge transformation with the parameter } \phi) \tag{3.2}$$

as follows:

$$\begin{aligned}
QU_\mu(x) &= i\psi_\mu(x)U_\mu(x)\,, \\
Q\psi_\mu(x) &= i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right), \\
Q\phi(x) &= 0\,, \\
Q\vec{\chi}(x) &= \vec{H}(x)\,, \qquad Q\vec{H}(x) = [\phi(x),\,\vec{\chi}(x)]\,, \\
Q\bar{\phi}(x) &= \eta(x)\,, \qquad Q\eta(x) = [\phi(x),\,\bar{\phi}(x)]\,.
\end{aligned} \tag{3.3}$$

Also, $QU_\mu(x)^\dagger = -iU_\mu(x)^\dagger\psi_\mu(x)$ follows from $U_\mu(x)U_\mu(x)^\dagger = 1$. All transformations except $QU_\mu(x)$ and $Q\psi_\mu(x)$ are of the same form as in the continuum case. Since (3.2) means

$$\begin{aligned}
Q^2U_\mu(x) &= i(Q\psi_\mu(x))\,U_\mu(x) - i\psi_\mu(x)\,(QU_\mu(x)) \\
&= \phi(x)U_\mu(x) - U_\mu(x)\phi(x+\hat{\mu})\,,
\end{aligned} \tag{3.4}$$

if we assume the formula "$QU_\mu(x) = \cdots$", the transformation $Q\psi_\mu(x)$ is determined.[5] Then, happily $Q^2\psi_\mu(x) = [\phi(x),\,\psi_\mu(x)]$ is satisfied, and the Q-transformation is consistently closed. Note that we use the dimensionless variables here, and that various quantities are of the following orders:

$$\begin{aligned}
\psi_\mu(x), \vec{\chi}(x), \eta(x) &= O(a^{3/2})\,, \qquad \phi(x), \bar{\phi}(x) = O(a)\,, \qquad \vec{H}(x) = O(a^2)\,, \\
Q &= O(a^{1/2})\,.
\end{aligned} \tag{3.5}$$

The first term in the r.h.s. of "$Q\psi_\mu(x) = \cdots$" in (3.3) is of subleading order $O(a^3)$.

3.1 Lattice action

Once we have the Q-transformation rule closed among lattice variables, it is almost straightforward to construct the lattice action with the exact supersymmetry Q:

$$\begin{aligned}
S^{\mathrm{LAT}}_{\mathcal{N}=2} = Q\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[&\frac{1}{4}\eta(x)\,[\phi(x),\,\bar{\phi}(x)] - i\vec{\chi}(x)\cdot\vec{\Phi}(x) + \vec{\chi}(x)\cdot\vec{H}(x) + \\
&+ i\sum_{\mu=1}^{d}\psi_\mu(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)\Bigg]\,,
\end{aligned} \tag{3.6}$$

[5] The first term in the r.h.s. does not vanish because $i\psi_\mu(x)\psi_\mu(x) = -\frac{1}{2}f^{abc}\psi^a_\mu(x)\psi^b_\mu(x)T^c$ with f^{abc} being structure constants of the gauge group.

where

$$\Phi(x) = -i\,[U_{12}(x) - U_{21}(x)] \qquad \text{for } d=2\,, \tag{3.7}$$

$$\Phi_{\mathtt{A}}(x) = -i\left[U_{\mathtt{A}4}(x) - U_{4\mathtt{A}}(x) + \frac{1}{2}\sum_{\mathtt{B},\mathtt{C}=1}^{3} \varepsilon_{\mathtt{ABC}}\,(U_{\mathtt{BC}}(x) - U_{\mathtt{CB}}(x))\right] \qquad \text{for } d=4\,, \tag{3.8}$$

$$\Phi_{\mathtt{A}}(x) = -i\left[U_{\mathtt{A}8}(x) - U_{8\mathtt{A}}(x) + \frac{1}{2}\sum_{\mathtt{B},\mathtt{C}=1}^{7} c_{\mathtt{ABC}}\,(U_{\mathtt{BC}}(x) - U_{\mathtt{CB}}(x))\right] \qquad \text{for } d=8\,. \tag{3.9}$$

$U_{\mu\nu}(x)$ are plaquette variables written as

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+\hat\mu)U_\mu(x+\hat\nu)^\dagger U_\nu(x)^\dagger\,. \tag{3.10}$$

The action (3.6) is clearly Q-invariant from its Q-exact form. After acting Q in the r.h.s., the action takes the form

$$\begin{aligned} S^{\rm LAT}_{\mathcal{N}=2} = \frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[&\frac{1}{4}[\phi(x),\,\bar\phi(x)]^2 + \vec H(x)\cdot\vec H(x) - i\vec H(x)\cdot\vec\Phi(x) + \\ &+\sum_{\mu=1}^{d}\left(\phi(x) - U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right)\left(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right) - \\ &-\frac{1}{4}\eta(x)[\phi(x),\,\eta(x)] - \vec\chi(x)\cdot[\phi(x),\,\vec\chi(x)] - \\ &-\sum_{\mu=1}^{d}\psi_\mu(x)\psi_\mu(x)\left(\bar\phi(x) + U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right) + \\ &+i\vec\chi(x)\cdot Q\vec\Phi(x) - i\sum_{\mu=1}^{d}\psi_\mu(x)\left(\eta(x) - U_\mu(x)\eta(x+\hat\mu)U_\mu(x)^\dagger\right)\Bigg]. \end{aligned} \tag{3.11}$$

After integrating out $\vec H(x)$, induced $\vec\Phi(x)^2$ term yields the gauge kinetic term as the form

$$\frac{1}{2g_0^2}\sum_x\sum_{\mu<\nu}\mathrm{tr}\left[-(U_{\mu\nu}(x) - U_{\nu\mu}(x))^2\right] \tag{3.12}$$

which leads double winding plaquette terms. Note difference from the standard Wilson action

$$\frac{1}{2g_0^2}\sum_x\sum_{\mu<\nu}\mathrm{tr}\,[2 - U_{\mu\nu}(x) - U_{\nu\mu}(x)]\,. \tag{3.13}$$

Also, for $d=4,8$, there appear terms corresponding to the total derivatives (2.15).

In contrast with (3.13) giving the unique minimum $U_{\mu\nu}(x)=1$, the action (3.12) has many classical vacua

$$U_{\mu\nu}(x) = \begin{pmatrix} \pm1 & & \\ & \ddots & \\ & & \pm1 \end{pmatrix} \tag{3.14}$$

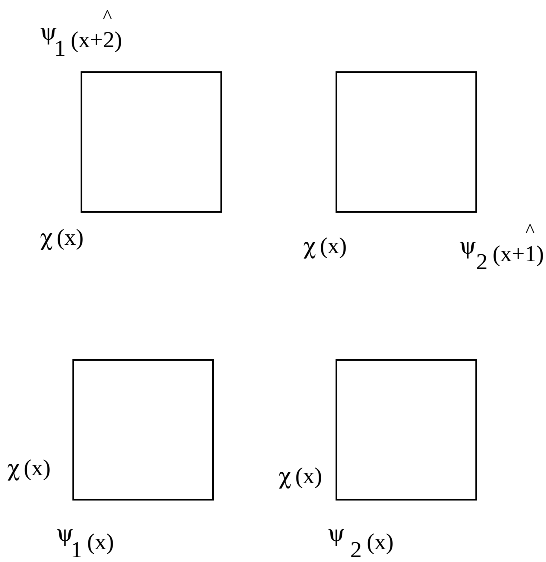

Figure 1: The plaquette structure of the term $\vec{\chi}(x) \cdot Q\vec{\Phi}(x)$ for $d = 2$ case. The variable U_μ is assigned on each link. Fermions χ and ψ_μ are distributed in the depicted four patterns.

up to gauge transformations, where any combinations of ± 1 with '-1' appearing even times are allowed in the diagonal entries. The number of '-1' means strength of a color flux through the plaquette. Since the configurations (3.14) can be taken freely for each plaquette, it leads a huge degeneracy of vacua with the number growing as exponential of the number of plaquettes. In order to see the dynamics of the model, we need to sum up contributions from all the minima, and the ordinary weak field expansion around a single vacuum $U_{\mu\nu}(x) = 1$ can not be justified.[6] Thus, we can not say anything on the continuum limit of the lattice model (3.11) without its nonperturbative investigations.

Aiming to resolve the degeneracy, we add the following term to the action:

$$\Delta S = \frac{1}{2g_0^2}\,\rho \sum_x \sum_{\mu<\nu} \mathrm{tr}\,(2 - U_{\mu\nu}(x) - U_{\nu\mu}(x)) \tag{3.15}$$

which is proportional to the standard Wilson action (3.13), resolving the degeneracy with the split $\frac{4\rho}{g_0^2}$. We will take the coefficient ρ so that ΔS vanishes in the classical continuum limit but then the split grows infinite. For example, in case of the two-dimensional $M \times M$ periodic lattice, the continuum limit is taken as $a \to 0$ with $g_2^2 \equiv g_0^2 a^{-2}$ and $L \equiv Ma$ fixed. ρ can be chosen as $\rho = \frac{1}{M^s} = \frac{a^s}{L^s}$ with the parameter $0 < s < 2$. Since the number of the lifted vacua with $\Delta S = \frac{4\rho}{g_0^2} = \frac{4}{g_2^2 L^s}\frac{1}{a^{2-s}}$ is proportional to the number of plaquettes, the entropy effect does not obstruct the decoupling. While this term justifies the expansion around $U_{\mu\nu}(x) = 1$, it breaks the supersymmetry Q. As is seen in section 3.3, however in the case of $d = 2$ it does not affect the renormalization argument.

The last two terms in (3.11) give rise to the fermion kinetic term. In particular, $\vec{\chi}(x) \cdot Q\vec{\Phi}(x)$ has the plaquette structure where two fermions sit at the same site or separately at the nearest neighbor sites. As an example, figure 1 shows the situation for $d = 2$ case.

[6]This is the same difficulty already encountered in ref. [4]. We thank Y. Shamir for the crucial comment.

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Owing to this structure, the lattice model does not have 90-degree rotational symmetry or the reflection positivity, either. However for $d = 2$ case the model is shown to flow to the desired continuum theory without any fine tuning of parameters.

3.2 Absence of fermion doubling

We expand the exponential of the link variable (3.1), and look at the kinetic terms in the action (3.11). Because in bosonic sector no species doublers appear, in fermionic sector also no doublers are expected due to the exact supersymmetry Q of (3.11). Let us see the fermionic sector explicitly.

After rescaling each fermion variable by $a^{3/2}$ as indicated in (3.5), the fermion kinetic terms are expressed as

$$\begin{aligned} S_f^{(2)} &= \frac{a^4}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\left[-\frac{1}{2}\Psi(x)^T(P_\mu+\gamma_\mu)\Delta_\mu\Psi(x)+\frac{1}{2}\Psi(x)^T(P_\mu-\gamma_\mu)\Delta_\mu^*\Psi(x)\right] \\ &= \frac{a^4}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\left[-\frac{1}{2}\Psi(x)^T\gamma_\mu(\Delta_\mu+\Delta_\mu^*)\Psi(x)-a\frac{1}{2}\Psi(x)^TP_\mu\Delta_\mu\Delta_\mu^*\Psi(x)\right], \end{aligned} \tag{3.16}$$

where fermions were combined as (2.13). P_μ are hermitian matrices satisfying $\{P_\mu, P_\nu\} = 2\delta_{\mu\nu}$. For their explicit form, see appendix B. Δ_μ and Δ_μ^* represent forward and backward difference operators respectively:

$$\Delta_\mu f(x) \equiv \frac{1}{a}\left(f(x+\hat{\mu})-f(x)\right), \qquad \Delta_\mu^* f(x) \equiv \frac{1}{a}\left(f(x)-f(x-\hat{\mu})\right). \tag{3.17}$$

The last term containing P_μ has a similar structure to the Wilson term, and plays a role of removing fermion doublers. The lattice action is $\mathrm{U}(1)_R$ invariant, and the fermion doublers are removed keeping the chiral $\mathrm{U}(1)_R$. Thus, the model must break some assumptions of Nielsen-Ninomiya's no go theorem [19]. In fact, broken is the assumption "*There exists a conserved charge Q_F corresponding to the fermion number.*". For simplicity, let us see the situation in $d = 2$ case. When combining fermions into a two-component Dirac spinor as

$$\zeta = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_1 - i\psi_2 \\ \chi + i\frac{1}{2}\eta\end{pmatrix}, \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}\left(\psi_1 + i\psi_2,\, \chi - i\frac{1}{2}\eta\right), \tag{3.18}$$

Q_F corresponds to the following $\mathrm{U}(1)_J$ rotation:

$$\zeta \to e^{i\theta}\zeta, \qquad \bar{\zeta} \to e^{-i\theta}\bar{\zeta}. \tag{3.19}$$

The first term in (3.16), giving a naive fermion kinetic term on the lattice, is written as the combination $\bar{\zeta}_\alpha\zeta_\beta$, which is invariant under the $\mathrm{U}(1)_J$. On the other hand, the last term containing P_μ takes the form:

$$\frac{a^4}{2g_0^2}\sum_x\frac{a}{2}\,\mathrm{tr}\left[\varepsilon_{\alpha\beta}\zeta_\alpha(\Delta_1\Delta_1^*+i\Delta_2\Delta_2^*)\zeta_\beta - \varepsilon_{\alpha\beta}\bar{\zeta}_\alpha(\Delta_1\Delta_1^*-i\Delta_2\Delta_2^*)\bar{\zeta}_\beta\right] \tag{3.20}$$

to break the $\mathrm{U}(1)_J$ invariance.

$p = a + b + 3c$	$\varphi^a \partial^b \psi^{2c}$
0	1
1	φ
2	φ^2
3	$\varphi^3, \quad \psi\psi, \quad \varphi\partial\varphi$
4	$\varphi^4, \quad \varphi^2\partial\varphi, \quad (\partial\varphi)^2, \quad \psi\partial\psi, \quad \varphi\psi\psi$

Table 1: List of operators with $p \leq 4$.

3.3 Renormalization

At the classical level, the lattice action (3.6) leads to the continuum action (2.7) in the limit $a \to 0$ with $g_d^{-2} \equiv a^{4-d} g_0^{-2}$ kept fixed, and thus the $\mathcal{N} = 2$ supersymmetry and rotational symmetry in d-dimensions are restored. We will check whether the symmetry restoration persists against quantum corrections, i.e. whether symmetry of the lattice action forbids any relevant or marginal operators induced which obstruct the symmetry restoration.

Here, we consider the case $d = 2$ only, where the $\mathrm{U}(1)_R$ symmetry is not anomalous. Since the model has the critical point $g_0 = 0$ from the asymptotic freedom, we shall consider the renormalization effect perturbatively. For a while, we discuss without the supersymmetry breaking term ΔS. The mass dimension of the coupling g_2^2 is two. For generic boson field φ (other than the auxiliary fields) and fermion field ψ, the dimensions are 1 and 3/2 respectively. Thus, operators of the type $\varphi^a \partial^b \psi^{2c}$ have the dimension $p \equiv a + b + 3c$, where '∂' means a derivative with respect to the coordinates. From dimensional analysis, the operators receive the following radiative corrections up to some powers of possible logarithmic factors:

$$\left(\frac{a^{p-4}}{g_2^2} + c_1 a^{p-2} + c_2 a^p g_2^2 + \cdots \right) \int \mathrm{d}^2 x \, \varphi^a \partial^b \psi^{2c} \,, \tag{3.21}$$

where $c_1, c_2, \ldots$ are constants dependent on N. The first, second and third terms in the parentheses represent the contributions at tree, one-loop and two-loop levels. It is easily seen from the fact that g_2^2 appears as an overall factor in front of the action and plays the same role as the Planck constant $\hbar$. Due to the super-renormalizable property in two-dimensional theory, the relevant corrections terminate at the two-loop. From the above formula, it is seen that the following operators can be relevant or marginal in the $a \to 0$ limit: operators with $p \leq 2$ induced at the one-loop level and with $p = 0$ at the two-loop level. Operators with $p \leq 4$ are listed in table 1.

Since the identity operator does not affect the spectrum, we have to check operators of the types φ and φ^2 only. Gauge symmetry and $\mathrm{U}(1)_R$ invariance allow the operator $\mathrm{tr}\, \phi\bar{\phi}$, while it is forbidden by the supersymmetry Q. Hence, no relevant or marginal operators except the identity are generated by radiative corrections.

When taking into account the term ΔS, we still have gauge invariance and $\mathrm{U}(1)_R$ symmetry but not the supersymmetry Q. This time, $\mathrm{tr}\, \phi\bar{\phi}$ might seem to remain. However, such is not the case. The supersymmetry breaking effect comes from loop diagrams containing the vertices of ΔS, and insertion of n ΔS-vertices yields the extra factor $\rho^n = a^{ns}/L^{ns}$

to each term in (3.21). So we can see that contribution from the supersymmetry breaking term to $\mathrm{tr}\,\phi\bar{\phi}$ becomes irrelevant behaving as a^{ns}. The conclusion of the renormalization argument without ΔS is not affected by including the breaking term. In the continuum limit, thus full supersymmetry and rotational symmetry are considered to be restored without any fine tuning.

4. 'Balanced topological field theory form' of $\mathcal{N} = 4$ super Yang-Mills theories

In this section, we consider $\mathcal{N} = 4$ SYM theories in two- and four-dimensions, which are obtained by further dimensional reduction from the $\mathcal{N} = 2$ theories in $d = 4,\ 8$, respectively. The theories can be written as exact forms with respect to both of two supercharges, which we refer as 'balanced topological field theory (BTFT) form' [13, 14, 15]. As the name 'balanced' means, fields carrying the positive $\mathrm{U}(1)_R$ charge are balanced with those carrying the negative charge, and thus the $\mathrm{U}(1)_R$ symmetry is not anomalous.

4.1 Two-dimensional case

After dimensional reduction with respect to the x^3 and x^4 directions, the $\mathcal{N} = 2$ SYM in $d = 4$ becomes two-dimensional $\mathcal{N} = 4$ SYM theory, which is written as the following 'BTFT form':

$$
\begin{aligned}
S^{2d}_{\mathcal{N}=4} &= Q_+Q_-\mathcal{F}^{2d}_{\mathcal{N}=4}\,,\\
\mathcal{F}^{2d}_{\mathcal{N}=4} &= \frac{1}{2g_2^2}\int \mathrm{d}^2x\,\mathrm{tr}\left[-iB\Phi - \psi_{+\mu}\psi_{-\mu} - \chi_+\chi_- - \frac{1}{4}\eta_+\eta_-\right],
\end{aligned}
\tag{4.1}
$$

where $Q_\pm$ are two of supercharges of the $\mathcal{N} = 4$ theory specified below, and $\Phi \equiv 2F_{12}$. Bosons are gauge fields A_μ ($\mu = 1,\ 2$) and scalar fields $B \equiv A_3$, $C \equiv 2A_4$, ϕ, $\bar{\phi}$. Also, there are auxiliary fields $\tilde{H}_\mu$, H. Scalars B and C appear in the process of the dimensional reduction. Other fields $\psi_{\pm\mu}$, $\chi_\pm$, $\eta_\pm$ are fermions. Transformation rule of the supersymmetry $Q_\pm$ is given by

$$
\begin{aligned}
&Q_+A_\mu = \psi_{+\mu}\,, \qquad Q_+\psi_{+\mu} = iD_\mu\phi\,, \qquad Q_-\psi_{+\mu} = \frac{i}{2}D_\mu C - \tilde{H}_\mu\,,\\
&Q_-A_\mu = \psi_{-\mu}\,, \qquad Q_-\psi_{-\mu} = -iD_\mu\bar{\phi}\,, \qquad Q_+\psi_{-\mu} = \frac{i}{2}D_\mu C + \tilde{H}_\mu\,,\\
&Q_+\tilde{H}_\mu = [\phi,\,\psi_{-\mu}] - \frac{1}{2}[C,\,\psi_{+\mu}] - \frac{i}{2}D_\mu\eta_+\,,\\
&Q_-\tilde{H}_\mu = [\bar{\phi},\,\psi_{+\mu}] + \frac{1}{2}[C,\,\psi_{-\mu}] + \frac{i}{2}D_\mu\eta_-\,,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&Q_+B = \chi_+\,, \qquad Q_+\chi_+ = [\phi,\,B]\,, \qquad Q_-\chi_+ = \frac{1}{2}[C,\,B] - H\,,\\
&Q_-B = \chi_-\,, \qquad Q_-\chi_- = -[\bar{\phi},\,B]\,, \qquad Q_+\chi_- = \frac{1}{2}[C,\,B] + H\,,\\
&Q_+H = [\phi,\,\chi_-] + \frac{1}{2}[B,\,\eta_+] - \frac{1}{2}[C,\,\chi_+]\,,\\
&Q_-H = [\bar{\phi},\,\chi_+] - \frac{1}{2}[B,\,\eta_-] + \frac{1}{2}[C,\,\chi_-]\,,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&Q_+C = \eta_+\,, \qquad Q_+\eta_+ = [\phi,\, C]\,, \qquad Q_-\eta_+ = -[\phi,\, \bar\phi]\,,\\
&Q_-C = \eta_-\,, \qquad Q_-\eta_- = -[\bar\phi,\, C]\,, \qquad Q_+\eta_- = [\phi,\, \bar\phi]\,,\\
&Q_+\phi = 0\,, \qquad Q_-\phi = -\eta_+\,, \qquad Q_+\bar\phi = \eta_-\,, \quad Q_-\bar\phi = 0\,.
\end{aligned} \tag{4.4}
$$

The transformation leads the following nilpotency of $Q_\pm$ (up to gauge transformations):

$$
\begin{aligned}
Q_+^2 &= (\text{infinitesimal gauge transformation with the parameter } \phi)\,,\\
Q_-^2 &= (\text{infinitesimal gauge transformation with the parameter } -\bar\phi)\,,\\
\{Q_+, Q_-\} &= (\text{infinitesimal gauge transformation with the parameter } C)\,.
\end{aligned} \tag{4.5}
$$

Since $\mathcal{F}^{2d}_{\mathcal{N}=4}$ is gauge invariant, clearly $S^{2d}_{\mathcal{N}=4}$ is invariant under the $Q_\pm$ transformations. Note that the form of $\mathcal{F}^{2d}_{\mathcal{N}=4}$ is not unique because the action is invariant under the shift

$$
\mathcal{F}^{2d}_{\mathcal{N}=4} \to \mathcal{F}^{2d}_{\mathcal{N}=4} + Q_+G_- + Q_-G_+ \tag{4.6}
$$

with gauge invariant $G_\pm$.

With the same γ-matrix convention as the $d = 4$ case in appendix A, notations for fermions in the 'TFT form' (2.13) correspond to

$$
\Psi^T = \left(\psi_{+1}, \psi_{+2}, \chi_+, \frac{1}{2}\eta_+, \psi_{-1}, \psi_{-2}, \chi_-, \frac{1}{2}\eta_-\right). \tag{4.7}
$$

Q_+ is nothing but the dimensional reduction of Q of the $\mathcal{N} = 2$, $d = 4$ case, and Q_- represents the transformation (2.6) with $\epsilon^T = (0, 0, 0, -\varepsilon, 0, 0, 0, 0)$. ε is real Grassmann.

Under the dimensional reduction, the $\mathrm{U}(1)_R$ internal symmetry enhances to $\mathrm{SU}(2)_R$, which is a subgroup of $\mathrm{SU}(4)$ internal symmetry of $\mathcal{N} = 4$ theory. The $\mathrm{SU}(2)_R$ generators are expressed as

$$
\begin{aligned}
J_{++} &= \int \mathrm{d}^2x \left[\psi^a_{+\mu}(x)\frac{\delta}{\delta\psi^a_{-\mu}(x)} + \chi^a_+(x)\frac{\delta}{\delta\chi^a_-(x)} - \eta^a_+(x)\frac{\delta}{\delta\eta^a_-(x)} + \right.\\
&\qquad \left. + 2\phi^a(x)\frac{\delta}{\delta C^a(x)} - C^a(x)\frac{\delta}{\delta\bar\phi^a(x)}\right],\\
J_{--} &= \int \mathrm{d}^2x \left[\psi^a_{-\mu}(x)\frac{\delta}{\delta\psi^a_{+\mu}(x)} + \chi^a_-(x)\frac{\delta}{\delta\chi^a_+(x)} - \eta^a_-(x)\frac{\delta}{\delta\eta^a_+(x)} - \right.\\
&\qquad \left. - 2\bar\phi^a(x)\frac{\delta}{\delta C^a(x)} + C^a(x)\frac{\delta}{\delta\phi^a(x)}\right],\\
J_0 &= \int \mathrm{d}^2x \left[\psi^a_{+\mu}(x)\frac{\delta}{\delta\psi^a_{+\mu}(x)} - \psi^a_{-\mu}(x)\frac{\delta}{\delta\psi^a_{-\mu}(x)} + \chi^a_+(x)\frac{\delta}{\delta\chi^a_+(x)} - \chi^a_-(x)\frac{\delta}{\delta\chi^a_-(x)} + \right.\\
&\qquad \left. + \eta^a_+(x)\frac{\delta}{\delta\eta^a_+(x)} - \eta^a_-(x)\frac{\delta}{\delta\eta^a_-(x)} + 2\phi^a(x)\frac{\delta}{\delta\phi^a(x)} - 2\bar\phi^a(x)\frac{\delta}{\delta\bar\phi^a(x)}\right],
\end{aligned} \tag{4.8}
$$

with 'a' being the index of a basis of the gauge group generators, and satisfy the algebra:

$$
[J_0,\, J_{++}] = 2J_{++}\,, \qquad [J_0,\, J_{--}] = -2J_{--}\,, \qquad [J_{++},\, J_{--}] = J_0\,. \tag{4.9}
$$

J_0 is a generator of the $\mathrm{U}(1)_R$ rotation, which is contained in $\mathrm{SU}(2)_R$ as its Cartan subalgebra. J_{++} (J_{--}) raises (lowers) the $\mathrm{U}(1)_R$ charge by two-units. For fermions, subscript

$\pm$ means the $\mathrm{U}(1)_R$ charge ± 1, and $Q_\pm$ raises/lowers the charge by one. For auxiliary fields $\tilde{H}_\mu$, H, the charge zero is assigned. Under the $\mathrm{SU}(2)_R$, each of $(\psi^a_{+\mu}, \psi^a_{-\mu})$, (χ^a_+, χ^a_-), $(\eta^a_+, -\eta^a_-)$ and (Q_+, Q_-) transforms as a doublet, and $(\phi^a, C^a, -\bar{\phi}^a)$ as a triplet.

The $Q_\pm$-transformation properties of (4.2), (4.3) and (4.4) are symbolically summarized as

$$\begin{array}{ccccc} & & \psi_{+\mu} & & \\ & \nearrow & & \searrow & \\ A_\mu & & & & \tilde{H}_\mu , \\ & \searrow & & \nearrow & \\ & & \psi_{-\mu} & & \end{array} \qquad \begin{array}{ccccc} & & \chi_+ & & \\ & \nearrow & & \searrow & \\ B & & & & H , \\ & \searrow & & \nearrow & \\ & & \chi_- & & \end{array} \qquad \begin{array}{ccc} \phi & & \\ & \searrow & \\ & & \eta_+ \\ & \nearrow & \\ C & & \\ & \searrow & \\ & & \eta_- \\ & \nearrow & \\ \bar{\phi} & & \end{array} \tag{4.10}$$

where $\nearrow$ and $\searrow$ represent the actions of Q_+ and Q_- respectively. The vertical direction indicates the $\mathrm{U}(1)_R$ charge (From the top, $+2$, $+1$, 0, -1, -2.).

Finally, let us note a symmetry of the action (4.1) under exchanging the two supercharges $Q_+ \leftrightarrow Q_-$ with

$$\begin{array}{lll} \phi \to -\bar{\phi} \,, & \bar{\phi} \to -\phi \,, & B \to -B \,, \\ \chi_+ \to -\chi_- \,, & \chi_- \to -\chi_+ \,, & \tilde{H}_\mu \to -\tilde{H}_\mu \,, \\ \psi_{\pm\mu} \to \psi_{\mp\mu} \,, & \eta_\pm \to \eta_\mp \,. & \end{array} \tag{4.11}$$

4.2 Four-dimensional case

Dimensional reduction from the $\mathcal{N} = 2$ theory in $d = 8$ with respect to the $x^5, \ldots, x^8$ directions leads four-dimensional $\mathcal{N} = 4$ SYM theory. Its 'BTFT form' is as follows:

$$\begin{aligned} S^{4d}_{\mathcal{N}=4} &= Q_+ Q_- \mathcal{F}^{4d}_{\mathcal{N}=4} \,, \qquad (4.12) \\ \mathcal{F}^{4d}_{\mathcal{N}=4} &= \frac{1}{2g_4^2} \int \mathrm{d}^4 x \, \mathrm{tr} \left[-i\vec{B} \cdot \vec{\Phi} - \frac{1}{3} \varepsilon_{\mathtt{ABC}} B_{\mathtt{A}} [B_{\mathtt{B}}, B_{\mathtt{C}}] - \psi_{+\mu}\psi_{-\mu} - \vec{\chi}_+ \cdot \vec{\chi}_- - \frac{1}{4}\eta_+\eta_- \right] , \end{aligned}$$

where $\mu = 1, \ldots, 4$, and $\vec{\Phi}$, $\vec{B}$, $\vec{\chi}_\pm$ are three-component vectors with the indices $\mathtt{A}, \mathtt{B}, \mathtt{C} = 1, 2, 3$. $\Phi_{\mathtt{A}}$ are functions of the field strengths $F_{\mu\nu}$: $\Phi_{\mathtt{A}} = 2\left(F_{\mathtt{A}4} + \frac{1}{2} \varepsilon_{\mathtt{ABC}} F_{\mathtt{BC}}\right)$, Bosonic field contents are gauge fields A_μ ($\mu = 1, \ldots, 4$) and scalar fields $B_1 \equiv -A_5$, $B_2 \equiv A_6$, $B_3 \equiv A_7$, $C \equiv 2A_8$, ϕ, $\bar{\phi}$. Also, auxiliary fields are $\tilde{H}_\mu$ and $H_{\mathtt{A}}$. Scalars $\vec{B}$, C arise via the dimensional reduction. Other fields $\psi_{\pm\mu}$, $\vec{\chi}_\pm$ and $\eta_\pm$ are fermionic. The field contents are almost same as the two-dimensional $\mathcal{N} = 4$ case except that each field in the quartet $(B, \chi_\pm, H)$ is replaced with a three-component vector. For fields other than the quartet $(\vec{B}, \vec{\chi}_\pm, \vec{H})$, the $Q_\pm$ transformations (4.2) and (4.4) do not change. The quartet transforms as

$$\begin{aligned} Q_+ \vec{B} &= \vec{\chi}_+ \,, \qquad & Q_+ \vec{\chi}_+ &= [\phi, \vec{B}] \,, \\ Q_- \vec{B} &= \vec{\chi}_- \,, \qquad & Q_- \vec{\chi}_- &= -[\bar{\phi}, \vec{B}] \,, \end{aligned}$$

$$
\begin{aligned}
Q_-\chi_{+\mathtt{A}} &= \frac{1}{2}[C,\, B_\mathtt{A}] - \frac{1}{2}\,\varepsilon_{\mathtt{ABC}}\,[B_\mathtt{B},\, B_\mathtt{C}] - H_\mathtt{A}\,, \\
Q_+\chi_{-\mathtt{A}} &= \frac{1}{2}[C,\, B_\mathtt{A}] + \frac{1}{2}\,\varepsilon_{\mathtt{ABC}}\,[B_\mathtt{B},\, B_\mathtt{C}] + H_\mathtt{A}\,, \\
Q_+H_\mathtt{A} &= [\phi,\, \chi_{-\mathtt{A}}] + \frac{1}{2}[B_\mathtt{A},\, \eta_+] - \frac{1}{2}[C,\, \chi_{+\mathtt{A}}] - \varepsilon_{\mathtt{ABC}}\,[B_\mathtt{B},\, \chi_{+\mathtt{C}}]\,, \\
Q_-H_\mathtt{A} &= [\bar\phi,\, \chi_{+\mathtt{A}}] - \frac{1}{2}[B_\mathtt{A},\, \eta_-] + \frac{1}{2}[C,\, \chi_{-\mathtt{A}}] - \varepsilon_{\mathtt{ABC}}\,[B_\mathtt{B},\, \chi_{-\mathtt{C}}]\,. \qquad (4.13)
\end{aligned}
$$

The nilpotency (4.5) is valid also here, and clearly seen is the $Q_\pm$ invariance of the action. With the same γ-matrix convention as in the $d = 8$ case in appendix A, fermions (2.13) correspond to

$$
\Psi^T = \left(\psi_{+1}, \ldots, \psi_{+4}, -\chi_{+1}, \chi_{+2}, \chi_{+3}, \frac{1}{2}\eta_+, \psi_{-1}, \ldots, \psi_{-4}, -\chi_{-1}, \chi_{-2}, \chi_{-3}, \frac{1}{2}\eta_-\right). \qquad (4.14)
$$

Q_+ is nothing but the dimensional reduction of Q for the $\mathcal{N} = 2$, $d = 8$ case, and Q_- represents the transformation (2.6) with $\epsilon^T = (0, 0, 0, 0, 0, 0, 0, -\varepsilon, 0, 0, 0, 0, 0, 0, 0, 0)$.

Enhancement of $\mathrm{U}(1)_R$ to $\mathrm{SU}(2)_R$ takes place, and the generators have the same form as in (4.8) with the straightforward modifications: $\mathrm{d}^2x \to \mathrm{d}^4x$ and $\chi^a_\pm(x) \to \chi^a_{\pm\mathtt{A}}(x)$. Also, the action (4.12) is invariant under $Q_+ \leftrightarrow Q_-$ with

$$
\begin{array}{lll}
\phi \to -\bar\phi\,, & \bar\phi \to -\phi\,, & \vec{B} \to -\vec{B}\,, \\
\vec\chi_+ \to -\vec\chi_-\,, & \vec\chi_- \to -\vec\chi_+\,, & \tilde{H}_\mu \to -\tilde{H}_\mu\,, \\
\psi_{\pm\mu} \to \psi_{\mp\mu}\,, & \eta_\pm \to \eta_\mp\,. &
\end{array} \qquad (4.15)
$$

5. Lattice formulation with two exact supercharges

Similarly to the $\mathcal{N} = 2$ cases, we consider to define the theories (4.1), (4.12) on hypercubic lattices preserving the two supercharges $Q_\pm$. Gauge fields A_μ are represented by the unitary variables $U_\mu(x)$ on the link $(x, x + \hat\mu)$, while the other fields are put on sites.

For both cases of two- and four-dimensions, the transformation rule (4.2) is modified as

$$
\begin{aligned}
Q_+U_\mu(x) &= i\psi_{+\mu}(x)U_\mu(x)\,, \\
Q_-U_\mu(x) &= i\psi_{-\mu}(x)U_\mu(x)\,, \\
Q_+\psi_{+\mu}(x) &= i\psi_{+\mu}\psi_{+\mu}(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right), \\
Q_-\psi_{-\mu}(x) &= i\psi_{-\mu}\psi_{-\mu}(x) + i\left(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U_\mu(x)^\dagger\right), \\
Q_-\psi_{+\mu}(x) &= \frac{i}{2}\left\{\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)C(x+\hat\mu)U_\mu(x)^\dagger\right) - \tilde{H}_\mu(x)\,, \\
Q_+\psi_{-\mu}(x) &= \frac{i}{2}\left\{\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)C(x+\hat\mu)U_\mu(x)^\dagger\right) + \tilde{H}_\mu(x)\,, \\
Q_+\tilde{H}_\mu(x) &= -\frac{1}{2}\left[\psi_{-\mu}(x),\, \phi(x) + U_\mu(x)\phi(x+\hat\mu)U_\mu(x)^\dagger\right] + \\
&\quad +\frac{1}{4}\left[\psi_{+\mu}(x),\, C(x) + U_\mu(x)C(x+\hat\mu)U_\mu(x)^\dagger\right] + \\
&\quad +\frac{i}{2}\left(\eta_+(x) - U_\mu(x)\eta_+(x+\hat\mu)U_\mu(x)^\dagger\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{2}\left[\psi_{+\mu}(x),\, \tilde{H}_\mu(x)\right] + \frac{1}{4}\left[\psi_{+\mu}(x)\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right], \\
Q_-\tilde{H}_\mu(x) \;=\; &-\frac{1}{2}\left[\psi_{+\mu}(x),\, \bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right] - \\
&-\frac{1}{4}\left[\psi_{-\mu}(x),\, C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right] - \\
&-\frac{i}{2}\left(\eta_-(x) - U_\mu(x)\eta_-(x+\hat{\mu})U_\mu(x)^\dagger\right) + \\
&+\frac{i}{2}\left[\psi_{-\mu}(x),\, \tilde{H}_\mu(x)\right] - \frac{1}{4}\left[\psi_{-\mu}(x)\psi_{-\mu}(x),\, \psi_{+\mu}(x)\right]. \qquad (5.1)
\end{aligned}
$$

The other transformations ((4.3), (4.4) in two-dimensions, (4.13), (4.4) in four-dimensions) do not change the form under the latticization. Note that this modification keeps the nilpotency (4.5).

Making use of the $Q_\pm$-transformation rule in terms of lattice variables, it is almost straightforward to construct lattice actions with the exact supercharges $Q_\pm$ for both of two- and four-dimensional cases:

$$
\begin{aligned}
S^{2d\,\mathrm{LAT}}_{\mathcal{N}=4} \;=\; & Q_+Q_-\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[-iB(x)\Phi(x) - \sum_{\mu=1}^{2}\psi_{+\mu}(x)\psi_{-\mu}(x) - \\
&\qquad - \chi_+(x)\chi_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\Bigg], \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
S^{4d\,\mathrm{LAT}}_{\mathcal{N}=4} \;=\; & Q_+Q_-\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[-i\vec{B}(x)\cdot\vec{\Phi}(x) - \frac{1}{3}\sum_{\mathtt{A},\mathtt{B},\mathtt{C}=1}^{3}\varepsilon_{\mathtt{ABC}}\,B_{\mathtt{A}}(x)\left[B_{\mathtt{B}}(x),\, B_{\mathtt{C}}(x)\right] - \\
&\qquad - \sum_{\mu=1}^{4}\psi_{+\mu}(x)\psi_{-\mu}(x) - \vec{\chi}_+(x)\cdot\vec{\chi}_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\Bigg], \qquad (5.3)
\end{aligned}
$$

where $\Phi(x)$ and $\vec{\Phi}(x)$ are given by (3.7) and (3.8), respectively. The orders of operators on the lattice are

$$
\begin{aligned}
\vec{B}(x), C(x), \phi(x), \bar{\phi}(x) &= O(a)\,, & \tilde{H}_\mu(x), \vec{H}(x) &= O(a^2)\,, \\
(\text{fermionic fields}) &= O(a^{3/2})\,, & Q_\pm &= O(a^{1/2})\,. \qquad (5.4)
\end{aligned}
$$

Note that the lattice formulation retains the symmetries under $\mathrm{SU}(2)_R$ as well as the $Q_+ \leftrightarrow Q_-$ exchange (4.11 or 4.15). Generators of $\mathrm{SU}(2)_R$ take the same form as the continuum case under the trivial modification accompanied with the latticization ($\int \mathrm{d}x \to \sum_x$, etc). These symmetries are used later in argument for the renormalization.

5.1 Two-dimensional case

Let us look closer the two-dimensional case (5.2), and discuss the renormalization.

After acting $Q_\pm$, the action takes the form:

$$
S^{2d\,\mathrm{LAT}}_{\mathcal{N}=4} = \frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[-i\left(\frac{1}{2}[C(x),\, B(x)] + H(x)\right)\Phi(x) + H(x)^2 +
$$

$$
\begin{aligned}
&+ i\chi_-(x)Q_+\Phi(x) - i\chi_+(x)Q_-\Phi(x) - iB(x)Q_+Q_-\Phi(x) - \\
&- [\phi(x),\, B(x)][\bar{\phi}(x),\, B(x)] - \frac{1}{4}[C(x),\, B(x)]^2 + \\
&+ \chi_+(x)[\bar{\phi}(x),\, \chi_+(x)] - \chi_-(x)[\phi(x),\, \chi_-(x)] + \chi_-(x)[C(x),\, \chi_+(x)] - \\
&- \chi_-(x)[B(x),\, \eta_+(x)] - \chi_+(x)[B(x),\, \eta_-(x)] - \\
&- \frac{1}{4}[\phi(x),\, \bar{\phi}(x)]^2 - \frac{1}{4}[\phi(x),\, C(x)][\bar{\phi}(x),\, C(x)] - \\
&- \frac{1}{4}\eta_-(x)[\phi(x),\, \eta_-(x)] + \frac{1}{4}\eta_+(x)[\bar{\phi}(x),\, \eta_+(x)] - \\
&- \frac{1}{4}\eta_+(x)[C(x),\, \eta_-(x)]\Big] + \\
+\frac{1}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\Big[&\tilde{H}_\mu(x)^2 - \frac{1}{2}\psi_{+\mu}(x)\psi_{+\mu}(x)\psi_{-\mu}(x)\psi_{-\mu}(x) + \\
&+ \left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right) \times \\
&\times \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) + \\
&+ \frac{1}{4}\left(C(x) - U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right)^2 - \\
&- \psi_{+\mu}(x)\psi_{+\mu}(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) + \\
&+ \psi_{-\mu}(x)\psi_{-\mu}(x)\left(\phi(x) + U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right) - \\
&- i\psi_{+\mu}(x)\left(\eta_-(x) - U_\mu(x)\eta_-(x+\hat{\mu})U_\mu(x)^\dagger\right) - \\
&- i\psi_{-\mu}(x)\left(\eta_+(x) - U_\mu(x)\eta_+(x+\hat{\mu})U_\mu(x)^\dagger\right) - \\
&- \frac{1}{2}\left\{\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right\}\left(C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right)\Big]. \qquad (5.5)
\end{aligned}
$$

Gauge kinetic terms appear from the $\Phi(x)^2$ term induced after $H(x)$ integrated out, whose structure is same as the $\mathcal{N}=2$ case. We add the supersymmetry breaking term (3.15) to single out the vacuum configuration $U_{\mu\nu}(x)=1$. $\tilde{H}_\mu(x)$ integrals yield kinetic terms of $B(x)$. Fermion kinetic terms come from the terms $\chi_\pm(x)Q_\mp\Phi(x)$ and $\psi_{\pm\mu}(x)(\eta_\mp(x) - U_\mu(x)\eta_\mp(x+\hat{\mu})U_\mu(x)^\dagger)$. The former has the same plaquette structure as in figure 1, and the species doublers are removed by the same mechanism.[7]

We repeat the same renormalization argument as in the $\mathcal{N}=2$, $d=2$ case. The symmetry of the lattice action is sufficient to restore full supersymmetry and rotational invariance in the continuum limit. For instance, gauge invariance and $\mathrm{SU}(2)_R$ symmetry allow the operators $\mathrm{tr}(4\phi\bar{\phi}+C^2)$ and $\mathrm{tr}\,B^2$, but they are not admissible from the supersymmetry $Q_\pm$. For operators of the type φ^2, supersymmetry breaking effect by ΔS is irrelevant in two-dimensions. Thus, radiative corrections are not allowed to generate any relevant or marginal operators except the identity, which means that full supersymmetry and rotational invariance are restored in the continuum limit.

[7]The fermion kinetic terms take the same form as (3.16) with P_μ given by (B.5), (B.6).

5.2 Four-dimensional case

For the four-dimensional case, the plaquette structure of fermion kinetic terms is same, and fermion doublers are removed.[8] Although it is necessary to add some term like (3.15) in order to resolve the vacuum degeneracy, we will not discuss it here. In this subsection, we shall simply see how the exact supersymmetry in the action (5.3) works to restrict radiative corrections. The action (5.3) has a symmetry under the transformation $x \equiv (x_1, x_2, x_3, x_4) \to \tilde{x} \equiv (x_3, x_1, x_2, x_4)$ (the cyclic permutation of x_1, x_2, x_3 with x_4 fixed) as

$$\begin{aligned}
(U_1(x), U_2(x), U_3(x), U_4(x)) &\to (U_2(\tilde{x}), U_3(\tilde{x}), U_1(\tilde{x}), U_4(\tilde{x}))\,, \\
(\psi_{\pm 1}(x), \psi_{\pm 2}(x), \psi_{\pm 3}(x), \psi_{\pm 4}(x)) &\to (\psi_{\pm 2}(\tilde{x}), \psi_{\pm 3}(\tilde{x}), \psi_{\pm 1}(\tilde{x}), \psi_{\pm 4}(\tilde{x}))\,, \\
(\tilde{H}_1(x), \tilde{H}_2(x), \tilde{H}_3(x), \tilde{H}_4(x)) &\to (\tilde{H}_2(\tilde{x}), \tilde{H}_3(\tilde{x}), \tilde{H}_1(\tilde{x}), \tilde{H}_4(\tilde{x}))\,, \\
(B_1(x), B_2(x), B_3(x)) &\to (B_2(\tilde{x}), B_3(\tilde{x}), B_1(\tilde{x}))\,, \\
(\chi_{\pm 1}(x), \chi_{\pm 2}(x), \chi_{\pm 3}(x)) &\to (\chi_{\pm 2}(\tilde{x}), \chi_{\pm 3}(\tilde{x}), \chi_{\pm 1}(\tilde{x}))\,, \\
(H_1(x), H_2(x), H_3(x)) &\to (H_2(\tilde{x}), H_3(\tilde{x}), H_1(\tilde{x}))\,, \\
(\phi(x), C(x), \bar{\phi}(x)) &\to (\phi(\tilde{x}), C(\tilde{x}), \bar{\phi}(\tilde{x}))\,, \\
\eta_\pm(x) &\to \eta_\pm(\tilde{x})\,.
\end{aligned} \tag{5.6}$$

The cyclic permutation of x_2, x_3, x_4 with x_1 fixed: $x \equiv (x_1, x_2, x_3, x_4) \to \tilde{x} \equiv (x_1, x_4, x_2, x_3)$ also keeps the action invariant:

$$\begin{aligned}
(U_1(x), U_2(x), U_3(x), U_4(x)) &\to (U_1(\tilde{x}), U_3(\tilde{x}), U_4(\tilde{x}), U_2(\tilde{x}))\,, \\
(\psi_{\pm 1}(x), \psi_{\pm 2}(x), \psi_{\pm 3}(x), \psi_{\pm 4}(x)) &\to (\psi_{\pm 1}(\tilde{x}), \psi_{\pm 3}(\tilde{x}), \psi_{\pm 4}(\tilde{x}), \psi_{\pm 2}(\tilde{x}))\,, \\
(\tilde{H}_1(x), \tilde{H}_2(x), \tilde{H}_3(x), \tilde{H}_4(x)) &\to (\tilde{H}_1(\tilde{x}), \tilde{H}_3(\tilde{x}), \tilde{H}_4(\tilde{x}), \tilde{H}_2(\tilde{x}))\,, \\
(B_1(x), B_2(x), B_3(x)) &\to (B_3(\tilde{x}), -B_1(\tilde{x}), -B_2(\tilde{x}))\,, \\
(\chi_{\pm 1}(x), \chi_{\pm 2}(x), \chi_{\pm 3}(x)) &\to (\chi_{\pm 3}(\tilde{x}), -\chi_{\pm 1}(\tilde{x}), -\chi_{\pm 2}(\tilde{x}))\,, \\
(H_1(x), H_2(x), H_3(x)) &\to (H_3(\tilde{x}), -H_1(\tilde{x}), -H_2(\tilde{x}))\,, \\
(\phi(x), C(x), \bar{\phi}(x)) &\to (\phi(\tilde{x}), C(\tilde{x}), \bar{\phi}(\tilde{x}))\,, \\
\eta_\pm(x) &\to \eta_\pm(\tilde{x})\,.
\end{aligned} \tag{5.7}$$

Thus, the action does not change under arbitrary permutations generated by (5.6) and (5.7), which contain $x \equiv (x_1, x_2, x_3, x_4) \to \tilde{x} \equiv (x_4, x_2, x_1, x_3)$ (x_2: fixed) and $x \equiv (x_1, x_2, x_3, x_4) \to \tilde{x} \equiv (x_4, x_1, x_3, x_2)$ (x_3: fixed).

In analysis on the renormalization, since the mass dimension of the coupling g_4^2 is zero, we should consider operators with $p \leq 4$ at all loop levels. For $p = 1, 2$, there exist no possible operators due to the gauge invariance, $\mathrm{SU}(2)_R$ and $Q_\pm$ supersymmetry. In $p = 3$ case, it is convenient to use the result of $Q_\pm$-cohomology in appendix C. According to the result (C.13), any gauge invariant and $\mathrm{SU}(2)_R$ invariant operators α annihilated by $Q_\pm$ can be written as the form:

$$\alpha = \alpha_0 + Q_+\beta_- + Q_+Q_-\gamma \tag{5.8}$$

[8] For the explicit form of the matrices P_μ, see (B.5), (B.7).

with α_0 proportional to the identity operator. From dimensional analysis and the degrees $(0,1)$ of β_-, the only possibility of the form of β_- is $A_\mu\psi_{-\nu}$. Since it can not be gauge invariant, the combination $Q_+\beta_-$ does not exist. While the $Q_+ \leftrightarrow Q_-$ symmetry requires γ being odd ($\gamma \to -\gamma$), there is no such dimension-two operator with $\mathrm{SU}(2)_R$ invariance.

In the above, we saw that symmetries of the action (5.3) prevent the relevant operators with $p = 1, 2, 3$ from being induced by loop corrections. For $p = 4$ case, all the terms appearing in the action are marginal operators passing various symmetry requirements. The symmetries of the lattice action protect each structure of the following five terms against the quantum corrections:

$$\begin{aligned}
&Q_+Q_-\ \mathrm{tr}\left[-i\vec{B}\cdot\vec{\Phi}\right], \qquad Q_+Q_-\ \mathrm{tr}\left[\sum_{\mathtt{A,B,C}}\varepsilon_{\mathtt{ABC}}\,B_{\mathtt{A}}\left[B_{\mathtt{B}},B_{\mathtt{C}}\right]\right], \qquad Q_+Q_-\ \mathrm{tr}\left[\sum_\mu \psi_{+\mu}\psi_{-\mu}\right], \\
&Q_+Q_-\ \mathrm{tr}\left[\vec{\chi}_+\cdot\vec{\chi}_-\right], \qquad Q_+Q_-\ \mathrm{tr}[\eta_+\eta_-],
\end{aligned} \tag{5.9}$$

while relative weights among the five possibly would receive some corrections.

The subtlety on $p = 4$ operators is specific in the four-dimensions. In the next two subsections, we will consider theories in lower dimensions obtained by dimensional reduction from the theory (5.3), and see two-dimensional model flowing to the desired continuum theories without fine tuning.

5.3 $\mathcal{N} = 8$ in three-dimensions

Dimensional reduction with respect to the x^4 direction from the action (5.3) leads the lattice action for three-dimensional $\mathcal{N} = 8$ SYM. Here, the dimensional reduction simply means to neglect the dependence on x^4 for all lattice variables. For example, the variable $U_4(x_1, x_2, x_3, x_4)$ on the link $(x, x+\hat{4})$ is reduced to the site variable $V(x_1, x_2, x_3)$ satisfying $V(x)V(x)^\dagger = 1$. The form of the action (5.3) does not change, but now we should read

$$U_{\mathtt{A}4}(x) = U_{\mathtt{A}}(x)V(x+\hat{\mathtt{A}})U_{\mathtt{A}}(x)^\dagger V(x)^\dagger\,, \qquad U_{4\mathtt{A}}(x) = V(x)U_{\mathtt{A}}(x)V(x+\hat{\mathtt{A}})^\dagger U_{\mathtt{A}}(x)^\dagger\,. \tag{5.10}$$

Also, we add the supersymmetry breaking term

$$\Delta S = \frac{1}{2g_0^2}\,\rho\left[\sum_{\mathtt{A}<\mathtt{B}}\mathrm{tr}\left(2 - U_{\mathtt{AB}}(x) - U_{\mathtt{BA}}(x)\right) + \sum_{\mathtt{A}=1}^{3}\mathrm{tr}\left(2 - U_{\mathtt{A}4}(x) - U_{4\mathtt{A}}(x)\right)\right], \tag{5.11}$$

where we consider on the $M \times M \times M$ periodic lattice, and the naive continuum limit is taken as $a \to 0$ with $g_3^2 \equiv g_0^2/a$ and $L \equiv Ma$ fixed. ρ is chosen as $\rho = 1/M^s = a^s/L^s$ with $0 < s < 1$ from the requirement that ΔS vanishes in the continuum limit while the levels of lifted vacua blow up. Under the dimensional reduction, symmetries of the four-dimensional model ($\mathrm{SU}(2)_R$, $Q_+ \leftrightarrow Q_-$ symmetry (4.15) and the permutation (5.6)) persist in the three-dimensional model.

Now, let us go to the renormalization argument. We discuss on effect of the breaking term ΔS later. In three-dimensions, the mass dimension of the coupling g_3^2 is one.

Operators of the type $\varphi^a\partial^b\psi^{2c}$ with the dimension $p = a+b+3c$ receive loop corrections as

$$\left(\frac{a^{p-4}}{g_3^2} + c_1 a^{p-3} + c_2 a^{p-2} g_3^2 + c_3 a^{p-1} g_3^4 + c_4 a^p g_3^6 + \cdots\right) \int \mathrm{d}^3x\, \varphi^a\partial^b\psi^{2c}\,, \tag{5.12}$$

where the notations are similar as in (3.21). From the formula, we read that operators with $p \leq 4-\ell$ are relevant or marginal at the ℓ-loop level. We can use the list in table 1.

For operators with $p = 1,2$, gauge invariance, $\mathrm{SU}(2)_R$ and $Q_+ \leftrightarrow Q_-$ symmetry allow

$$\mathrm{tr}(4\phi\bar{\phi} + C^2)\,, \qquad \mathrm{tr}(A_4^2), \quad \mathrm{tr}(B_1^2 + B_2^2 + B_3^2)\,, \qquad \mathrm{tr}(B_1B_2 + B_2B_3 + B_3B_1)\,, \tag{5.13}$$

while all of them are forbidden by the $Q_\pm$ supersymmetry. In order to see the $p = 3$ operators, we use the result of $Q_\pm$-cohomology (C.13). From dimensional analysis, candidates for β_- are $\mathrm{tr}(A_4\psi_{-\mu})$, but they are not Q_- invariant. Hence there is no operator of the type $Q_+\beta_-$. Symmetries under $Q_+ \leftrightarrow Q_-$ and the permutation $(x_1,x_2,x_3) \to (x_3,x_1,x_2)$ reduce candidates for γ to $\mathrm{tr}((B_1+B_2+B_3)A_4)$ alone.

Loop diagrams with n vertices of ΔS inserted induce supersymmetry breaking effect, which is suppressed by the factor a^{ns}/L^{ns} for each loop contribution in (5.12). The power of 'a' for the $p = 2$ operators at the one-loop is a^{ns-1}, so the operators (5.13) remain relevant for $n = 1$ and can not be excluded. On the other hand, the $p = 3$ operators becomes irrelevant when ΔS-vertices inserted.

Therefore, the radiative corrections are allowed to generate the operator $Q_+Q_-\,\mathrm{tr}((B_1+B_2+B_3)A_4)$ and those in (5.13). In order to reach the desired supersymmetric continuum theory, we have to add counter terms for the operators and cancel them with the radiative corrections by tuning parameters.

5.4 $\mathcal{N} = 8$ in two-dimensions

As a result of further dimensional reduction with respect to x^3, we obtain a lattice theory for $\mathcal{N} = 8$ in two-dimensions. In addition to the site variable $V(x)$ coming from $U_4(x)$, there appears a new site variable $W(x)$, which arises from $U_3(x)$, satisfying $W(x)W(x)^\dagger = 1$. The form of the action (5.3) is still valid with

$$\begin{aligned} U_{\mu 4}(x) &= U_{4\mu}(x)^\dagger = U_\mu(x)V(x+\hat{\mu})U_\mu(x)^\dagger V(x)^\dagger\,, \\ U_{\mu 3}(x) &= U_{3\mu}(x)^\dagger = U_\mu(x)W(x+\hat{\mu})U_\mu(x)^\dagger W(x)^\dagger\,, \\ U_{34}(x) &= U_{43}(x)^\dagger = W(x)V(x)W(x)^\dagger V(x)^\dagger\,. \end{aligned} \tag{5.14}$$

Here, $\mu = 1,2$ for gauge variables. Similar to the previous cases, we consider the $M\times M$ periodic lattice and the supersymmetry breaking term

$$\Delta S = \frac{1}{2g_0^2}\,\rho\sum_x \sum_{1\leq k<l\leq 4} \mathrm{tr}\left(2 - U_{kl}(x) - U_{lk}(x)\right)\,, \tag{5.15}$$

with $\rho = \frac{1}{M^s} = \frac{a^s}{L^s}$ $(0 < s < 2)$.

From the same renormalization argument, it is seen that loop corrections are not allowed to generate any relevant or marginal operators except the identity, which means restoration of full supersymmetry and rotational invariance in the continuum limit.

6. Summary and discussions

In this paper, SU(N) super Yang-Mills theories with extended supersymmetry have been constructed on hyper-cubic lattices of various dimensions, keeping one or two supercharges exactly. Our lattice models have a huge degeneracy of the classical vacua, which does not allow the standard argument based on the lattice perturbation theory. In order to avoid the difficulty, we added to the action some supersymmetry breaking term, which becomes invisible in the naive continuum limit. Among the lattice models, the desired continuum theories are obtained without any fine tuning of parameters for the cases $\mathcal{N} = 2, 4, 8$ in two-dimensions. We have arrived at the conclusion checking that symmetries of the lattice action forbid radiative corrections to generate any relevant or marginal operators which prevent the lattice theory from flowing to the continuum theory. It is interesting to see the restoration of full supersymmetry and rotational invariance explicitly by computer simulation.

To resolve the vacuum degeneracy, it would be also possible to impose the admissibility condition on each plaquette variable:

$$||1 - U_{\mu\nu}(x)|| < \epsilon \tag{6.1}$$

similar to that for gauge fields coupled to the Ginsparg-Wilson fermions [20]. It breaks the supersymmetry picking up contribution from the boundary $||1 - U_{\mu\nu}(x)|| = \epsilon$, but the breaking effect would vanish in the weak coupling limit. It will be interesting to pursue the possibility and to try reformulating our models in terms of the Ginsparg-Wilson fermions.

It is also worth to be investigated the dynamics of our lattice actions without the supersymmetry breaking terms, nonperturbative understanding of which will be necessary to identify the corresponding continuum theories.

The lattice actions have no symmetry under discrete rotations by 90 degrees due to the specific couplings between fermions χ and ψ. It is possible to modify the couplings keeping the supersymmetry so that the theories are symmetric under the 90-degree rotations. At the same time, however doublers arise from both of bosonic and fermionic sectors, as expected from the manifest supersymmetry. It is interesting to identify the corresponding continuum theories for both cases with and without the supersymmetry breaking terms.

While we have focussed on pure Yang-Mills theories, it is possible to extend our formulation to the case of matter fields coupled, since the actions of $\mathcal{N} = 2, 4$ supersymmetric QCD are rewritten as 'TFT and BTFT forms' [21].

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A. Explicit form of γ_i for $\mathcal{N}=2$

Here, we present the explicit form of the matrices γ_i employed for 'TFT form' of $\mathcal{N}=2$ SYM theories. γ_i are written as the form

$$\gamma_i = -i\begin{pmatrix} 0 & \mu_i \\ \mu_i^T & 0 \end{pmatrix} \qquad \text{for } i=1,\dots,D-2\,, \qquad \gamma_{D-1} = i\sigma_3 \otimes \mathbf{1}_{D-2}\,, \tag{A.1}$$

where σ_1, σ_2, σ_3 denote the Pauli matrices, and the explicit forms of μ_i are as follows.

For $d=2$,

$$\mu_1 = \sigma_1\,, \qquad \mu_2 = -\sigma_3\,. \tag{A.2}$$

For $d=4$,

$$\mu_1 = \begin{pmatrix} & \sigma_1 \\ -i\sigma_2 & \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} & -\sigma_3 \\ \mathbf{1}_2 & \end{pmatrix}, \qquad \mu_3 = \begin{pmatrix} i\sigma_2 & \\ & \sigma_1 \end{pmatrix}, \qquad \mu_4 = \begin{pmatrix} -\mathbf{1}_2 & \\ & -\sigma_3 \end{pmatrix}. \tag{A.3}$$

For $d=8$,

$$\begin{aligned}
\mu_1 &= \begin{pmatrix} &&& \sigma_1 \\ && -\sigma_1 & \\ & \sigma_1 && \\ -i\sigma_2 &&& \end{pmatrix}, \quad
\mu_2 = \begin{pmatrix} &&& -\sigma_3 \\ && -\sigma_3 & \\ & \sigma_3 && \\ \mathbf{1}_2 &&& \end{pmatrix}, \quad
\mu_3 = \begin{pmatrix} && \sigma_1 & \\ &&& \sigma_1 \\ -\sigma_1 &&& \\ & -i\sigma_2 && \end{pmatrix}, \\
\mu_4 &= \begin{pmatrix} && \sigma_3 & \\ &&& -\sigma_3 \\ -\sigma_3 &&& \\ & \mathbf{1}_2 && \end{pmatrix}, \quad
\mu_5 = \begin{pmatrix} & -\sigma_1 && \\ \sigma_1 &&& \\ &&& \sigma_1 \\ && -i\sigma_2 & \end{pmatrix}, \quad
\mu_6 = \begin{pmatrix} & -\sigma_3 && \\ \sigma_3 &&& \\ &&& -\sigma_3 \\ && \mathbf{1}_2 & \end{pmatrix}, \\
\mu_7 &= \begin{pmatrix} i\sigma_2 &&& \\ & i\sigma_2 && \\ && i\sigma_2 & \\ &&& \sigma_1 \end{pmatrix}, \quad
\mu_8 = \begin{pmatrix} -\mathbf{1}_2 &&& \\ & -\mathbf{1}_2 && \\ && -\mathbf{1}_2 & \\ &&& -\sigma_3 \end{pmatrix}.
\end{aligned} \tag{A.4}$$

The cases $d=4,8$ correspond also to 'BTFT form' for $\mathcal{N}=4$ SYM theories in two- and four-dimensions, respectively.

B. Explicit form of P_μ for $\mathcal{N}=2,4$

For both cases $\mathcal{N}=2$ and $\mathcal{N}=4$, fermion kinetic terms in the lattice actions are expressed as the form (3.16). We present the explicit form of matrices P_μ appearing in the expressions.

In the $\mathcal{N}=2$ theories, P_μ are written as the form

$$P_\mu = \begin{pmatrix} 0 & \nu_\mu \\ \nu_\mu & 0 \end{pmatrix} = \sigma_1 \otimes \nu_\mu \qquad \text{for } \mu = 1,\dots,d-1\,, \qquad P_d = \sigma_2 \otimes \mathbf{1}_d\,. \tag{B.1}$$

For each case of $d=2$, 4, 8, ν_μ are given as follows.

For $d = 2$ case,

$$\nu_1 = \sigma_2 \,. \tag{B.2}$$

For $d = 4$ case,

$$\nu_1 = \sigma_1 \otimes \sigma_2 \,, \qquad \nu_2 = \sigma_2 \otimes \mathbf{1}_2 \,, \qquad \nu_3 = -\sigma_3 \otimes \sigma_2 \,. \tag{B.3}$$

For $d = 8$ case,

$$\begin{gathered}
\nu_1 = \begin{pmatrix} & & & \sigma_2 \\ & & -i\sigma_1 & \\ & i\sigma_1 & & \\ \sigma_2 & & & \end{pmatrix}, \quad
\nu_2 = \begin{pmatrix} & & & -i\mathbf{1}_2 \\ & & -i\sigma_3 & \\ & i\sigma_3 & & \\ i\mathbf{1}_2 & & & \end{pmatrix}, \quad
\nu_3 = \begin{pmatrix} & & i\sigma_1 & \\ & & & \sigma_2 \\ -i\sigma_1 & & & \\ & \sigma_2 & & \end{pmatrix}, \\
\nu_4 = \begin{pmatrix} & & i\sigma_3 & \\ & & & -i\mathbf{1}_2 \\ -i\sigma_3 & & & \\ & i\mathbf{1}_2 & & \end{pmatrix}, \quad
\nu_5 = \begin{pmatrix} & -i\sigma_1 & & \\ i\sigma_1 & & & \\ & & & \sigma_2 \\ & & \sigma_2 & \end{pmatrix}, \\
\nu_6 = \begin{pmatrix} & -i\sigma_3 & & \\ i\sigma_3 & & & \\ & & & -i\mathbf{1}_2 \\ & & i\mathbf{1}_2 & \end{pmatrix}, \quad
\nu_7 = \begin{pmatrix} -\sigma_2 & & & \\ & -\sigma_2 & & \\ & & -\sigma_2 & \\ & & & \sigma_2 \end{pmatrix}.
\end{gathered} \tag{B.4}$$

In the $\mathcal{N} = 4$ theories, P_μ are expressed as

$$P_\mu = -i \begin{pmatrix} 0 & \omega_\mu \\ -\omega_\mu^T & 0 \end{pmatrix}, \tag{B.5}$$

where the explicit forms of ω_μ are presented as below.

For two-dimensional case,

$$\omega_1 = \begin{pmatrix} & i\sigma_2 \\ -\sigma_1 & \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} & \mathbf{1}_2 \\ \sigma_3 & \end{pmatrix}. \tag{B.6}$$

For four-dimensional case,

$$\begin{gathered}
\omega_1 = \begin{pmatrix} & & & i\sigma_2 \\ & & \sigma_1 & \\ & \sigma_1 & & \\ -\sigma_1 & & & \end{pmatrix}, \quad
\omega_2 = \begin{pmatrix} & & & \mathbf{1}_2 \\ & & \sigma_3 & \\ & \sigma_3 & & \\ \sigma_3 & & & \end{pmatrix}, \quad
\omega_3 = \begin{pmatrix} & & -\sigma_1 & \\ & & & i\sigma_2 \\ -\sigma_1 & & & \\ & -\sigma_1 & & \end{pmatrix}, \\
\omega_4 = \begin{pmatrix} & & -\sigma_3 & \\ & & & \mathbf{1}_2 \\ -\sigma_3 & & & \\ & \sigma_3 & & \end{pmatrix}.
\end{gathered} \tag{B.7}$$

C. $Q_\pm$-cohomology on lattice

In this appendix, we discuss on the equivalent cohomology based on two supercharges Q_+, Q_- nilpotent up to gauge transformations in the theory (5.3).[9] The basic fact is given in Theorem 2.1 in ref. [14], and this gives a generalization in the context of lattice gauge theory.

First, we assign the degrees (q_+, q_-) to each variable as follows:

$$\begin{array}{rlrlrl} U_\mu &: (0,0) & \vec{B} &: (1,1) & \phi &: (2,0) \\ \psi_{+\mu} &: (1,0) & \vec{\chi}_+ &: (2,1) & \bar{\phi} &: (0,2) \\ \psi_{-\mu} &: (0,1) & \vec{\chi}_- &: (1,2) & C &: (1,1) \\ \tilde{H}_\mu &: (1,1) & \vec{H} &: (2,2) & \eta_+ &: (2,1) \\ & & & & \eta_- &: (1,2)\,. \end{array} \tag{C.1}$$

The charges $q_\pm$ are measured by the operators:

$$\begin{aligned} L_+ = \sum_{x,a} \Bigg[& \psi^a_{+\mu}(x)\frac{\partial}{\partial \psi^a_{+\mu}(x)} + \tilde{H}^a_\mu(x)\frac{\partial}{\partial \tilde{H}^a_\mu(x)} + B^a_{\mathtt{A}}(x)\frac{\partial}{\partial B^a_{\mathtt{A}}(x)} + 2\chi^a_{+\mathtt{A}}(x)\frac{\partial}{\partial \chi^a_{+\mathtt{A}}(x)} + \\ & + \chi^a_{-\mathtt{A}}(x)\frac{\partial}{\partial \chi^a_{-\mathtt{A}}(x)} + 2H^a_{\mathtt{A}}(x)\frac{\partial}{\partial H^a_{\mathtt{A}}(x)} + C^a(x)\frac{\partial}{\partial C^a(x)} + 2\phi^a(x)\frac{\partial}{\partial \phi^a(x)} + \\ & + 2\eta^a_+(x)\frac{\partial}{\partial \eta^a_+(x)} + \eta^a_-(x)\frac{\partial}{\partial \eta^a_-(x)} \Bigg], \\ L_- = \sum_{x,a} \Bigg[& \psi^a_{-\mu}(x)\frac{\partial}{\partial \psi^a_{-\mu}(x)} + \tilde{H}^a_\mu(x)\frac{\partial}{\partial \tilde{H}^a_\mu(x)} + B^a_{\mathtt{A}}(x)\frac{\partial}{\partial B^a_{\mathtt{A}}(x)} + 2\chi^a_{-\mathtt{A}}(x)\frac{\partial}{\partial \chi^a_{-\mathtt{A}}(x)} + \\ & + \chi^a_{+\mathtt{A}}(x)\frac{\partial}{\partial \chi^a_{+\mathtt{A}}(x)} + 2H^a_{\mathtt{A}}(x)\frac{\partial}{\partial H^a_{\mathtt{A}}(x)} + C^a(x)\frac{\partial}{\partial C^a(x)} + 2\bar{\phi}^a(x)\frac{\partial}{\partial \bar{\phi}^a(x)} + \\ & + 2\eta^a_-(x)\frac{\partial}{\partial \eta^a_-(x)} + \eta^a_+(x)\frac{\partial}{\partial \eta^a_+(x)} \Bigg], \end{aligned} \tag{C.2}$$

where 'a' is the index with respect to the gauge group generators, and the summation over the indices μ and $\mathtt{A}$ is assumed. The $\mathrm{U}(1)_R$ charge is given by $q_+ - q_-$ and $J_0 = L_+ - L_-$. $Q_\pm$ and $J_{\pm\pm}$ shift the degrees as

$$\begin{aligned} Q_+ &: (q_+, q_-) \to (q_+ + 1, q_-)\,, & Q_- &: (q_+, q_-) \to (q_+, q_- + 1)\,, \\ J_{++} &: (q_+, q_-) \to (q_+ + 1, q_- - 1)\,, & J_{--} &: (q_+, q_-) \to (q_+ - 1, q_- + 1)\,. \end{aligned} \tag{C.3}$$

Next, it is convenient for later argument to introduce the 'homotopy operators' $K_\pm$:

$$K_+ = -\sum_{x,a} \Bigg[\psi^a_{+\mu}(x)\frac{\partial}{\partial \tilde{H}^a_\mu(x)} + 2\chi^a_{+\mathtt{A}}(x)\frac{\partial}{\partial H^a_{\mathtt{A}}(x)} - B^a_{\mathtt{A}}(x)\frac{\partial}{\partial \chi^a_{-\mathtt{A}}(x)} +$$

[9] The argument is valid also for the two-dimensional case (5.2) with reading the three-component vectors $\vec{B}$, $\vec{\chi}_\pm$, $\vec{H}$ as the one-component B, $\chi_\pm$, H.

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$$
\begin{aligned}
&+ 2\phi^a(x)\frac{\partial}{\partial\eta_+^a(x)} - C^a(x)\frac{\partial}{\partial\eta_-^a(x)}\Bigg] , \\
K_- &= \sum_{x,a}\Bigg[\psi_{-\mu}^a(x)\frac{\partial}{\partial\tilde{H}_\mu^a(x)} + 2\chi_{-\mathtt{A}}^a(x)\frac{\partial}{\partial H_{\mathtt{A}}^a(x)} + B_{\mathtt{A}}^a(x)\frac{\partial}{\partial\chi_{+\mathtt{A}}^a(x)} + \\
&\qquad + 2\bar{\phi}^a(x)\frac{\partial}{\partial\eta_-^a(x)} + C^a(x)\frac{\partial}{\partial\eta_+^a(x)}\Bigg] ,
\end{aligned}
\tag{C.4}
$$

with changing the degrees as

$$
K_+ : (q_+, q_-) \to (q_+, q_- - 1) , \qquad K_- : (q_+, q_-) \to (q_+ - 1, q_-) . \tag{C.5}
$$

Since there is no operator with negative $q_\pm$, for any operator α with (q_+, q_-)

$$
\begin{aligned}
J_{++}\alpha &= K_+\alpha = 0 \qquad \text{if } q_- = 0 , \\
J_{--}\alpha &= K_-\alpha = 0 \qquad \text{if } q_+ = 0 .
\end{aligned}
\tag{C.6}
$$

After some calculation, it turns out that the following algebra holds:

$$
\begin{aligned}
\{Q_+, K_+\} &= -J_{++} , \qquad \{Q_-, K_+\} = L_+ , \\
\{Q_-, K_-\} &= -J_{--} , \qquad \{Q_+, K_-\} = L_- ,
\end{aligned}
\tag{C.7}
$$

which plays a key role for the argument on $Q_\pm$-cohomology below.

Suppose that α is a gauge invariant and $\mathrm{SU}(2)_R$ invariant operator with the degrees (q_+, q_-), and that $Q_\pm$ annihilate α: $Q_\pm\alpha = 0$. Since $K_-Q_+\alpha = 0$, making use of (C.7) leads

$$
\alpha = \begin{cases} Q_+\left(\dfrac{1}{q_-}K_-\alpha\right) & \text{for } q_+ \geq 1,\ q_- \geq 1 , \\ \qquad 0 & \text{for } q_+ = 0,\ q_- \geq 1 . \end{cases} \tag{C.8}
$$

When $q_- = 0$, Q_+-cohomology can be nontrivial. Similarly, starting with $K_+Q_-\alpha = 0$, we arrive at

$$
\alpha = \begin{cases} Q_-\left(\dfrac{1}{q_+}K_+\alpha\right) & \text{for } q_+ \geq 1,\ q_- \geq 1 , \\ \qquad 0 & \text{for } q_+ \geq 1,\ q_- = 0 . \end{cases} \tag{C.9}
$$

Nontrivial Q_--cohomology can appear when $q_+ = 0$.

Let us consider the case $q_+ \geq 1$, $q_- \geq 1$. Note that $K_-\alpha$ is annihilated by Q_- as a consequence of (C.7). Thus, by the same argument to (C.8),

$$
K_-\alpha = Q_-\left(\frac{1}{q_+ - 1}K_+K_-\alpha\right) \qquad \text{for } q_+ \geq 2,\ q_- \geq 1 . \tag{C.10}
$$

Plugging this into (C.8) leads

$$
\alpha = Q_+Q_-\left(\frac{1}{q_-(q_+ - 1)}K_+K_-\alpha\right) \qquad \text{for } q_+ \geq 2,\ q_- \geq 1 . \tag{C.11}
$$

From the similar consideration on $K_+\alpha$, we obtain

$$
\alpha = Q_-Q_+\left(\frac{1}{q_+(q_- - 1)}K_-K_+\alpha\right) \qquad \text{for } q_+ \geq 1,\ q_- \geq 2 . \tag{C.12}
$$

In the case $q_+ = q_- = 0$, α is a function of U_μ. But, from the condition of $Q_\pm$-closed, it must be proportional to the identity operator. Finally, when $q_+ = q_- = 1$, α is expressed as $\alpha = Q_+(K_-\alpha) = Q_-(K_+\alpha)$, where $K_\pm\alpha$ are $Q_\pm$-closed and expanded by $\psi_{\pm\mu}$, respectively.

The above result is summarized as the following statement:

Any gauge invariant and $\mathrm{SU}(2)_R$ *invariant operator* α *annihilated by* $Q_\pm$ *can be expressed as*

$$\begin{aligned}\alpha &= \alpha_0 + Q_+\beta_- + Q_+Q_-\gamma \\ &= \alpha_0 + Q_-\beta_+ + Q_+Q_-\gamma\,. \end{aligned} \tag{C.13}$$

α_0 *is proportional to the identity, and* $\beta_\pm$, γ *are gauge invariant.* β_+ *(*β_-*) is a* Q_+*- (*Q_-*-) closed operator of the degrees* $(1,0)$ *(*$(0,1)$*), and has nontrivial cohomology with respect to* Q_+ *(*Q_-*).*

The argument is valid also for lattice SYM theories obtained by dimensional reduction from (5.3), for instance $\mathcal{N} = 8$ theories in two- and three-dimensions.

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087], erratum *ibid.* **B430** (1994) 485; *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[2] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[3] T. Banks and P. Windey, *Supersymmetric lattice theories*, *Nucl. Phys.* **B 198** (1982) 226;
I. Ichinose, *Supersymmetric lattice gauge theory*, *Phys. Lett.* **B 122** (1983) 68;
J. Bartels and J.B. Bronzan, *Supersymmetry on a lattice*, *Phys. Rev.* **D 28** (1983) 818;
J. Bartels and G. Kramer, *A lattice version of the Wess-Zumino model*, *Z. Physik* **C 20** (1983) 159;
R. Nakayama and Y. Okada, *Supercurrent anomaly in lattice gauge theory*, *Phys. Lett.* **B 134** (1984) 241;
S. Nojiri, *Continuous 'translation' and supersymmetry on the lattice*, *Prog. Theor. Phys.* **74** (1985) 819;
G. Curci and G. Veneziano, *Supersymmetry and the lattice: a reconciliation?*, *Nucl. Phys.* **B 292** (1987) 555;
K. Fujikawa, *Supersymmetry on the lattice and the Leibniz rule*, *Nucl. Phys.* **B 636** (2002) 80 [hep-th/0205095].

[4] S. Elitzur, E. Rabinovici and A. Schwimmer, *Supersymmetric models on the lattice*, *Phys. Lett.* **B 119** (1982) 165.

[5] D.B. Kaplan, *Dynamical generation of supersymmetry*, *Phys. Lett.* **B 136** (1984) 162;
J. Nishimura, *Four-dimensional $N = 1$ supersymmetric Yang-Mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B 406** (1997) 215 [hep-lat/9701013];

JHEP01(2004)015

N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A 13** (1998) 2841 [hep-th/9705152];
H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [hep-lat/9710089];
D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543–550 [hep-lat/0002030];
G.T. Fleming, J.B. Kogut and P.M. Vranas, *Super Yang-Mills on the lattice with domain wall fermions*, *Phys. Rev.* **D 64** (2001) 034510 [hep-lat/0008009];
I. Montvay, *Supersymmetric Yang-Mills theory on the lattice*, *Int. J. Mod. Phys.* **A 17** (2002) 2377 [hep-lat/0112007].

[6] S. Cecotti and L. Girardello, *Stochastic processes in lattice (extended) supersymmetry*, *Nucl. Phys.* **B 226** (1983) 417;
N. Sakai and M. Sakamoto, *Lattice supersymmetry and the Nicolai mapping*, *Nucl. Phys.* **B 229** (1983) 173;
S. Catterall and S. Karamov, *Exact lattice supersymmetry: the two-dimensional $N = 2$ Wess-Zumino model*, *Phys. Rev.* **D 65** (2002) 094501 [hep-lat/0108024];
Y. Kikukawa and Y. Nakayama, *Nicolai mapping vs. exact chiral symmetry on the lattice*, *Phys. Rev.* **D 66** (2002) 094508 [hep-lat/0207013];
K. Fujikawa, *$N = 2$ Wess-Zumino model on the $D = 2$ euclidean lattice*, *Phys. Rev.* **D 66** (2002) 074510 [hep-lat/0208015].

[7] D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a spatial lattice*, *J. High Energy Phys.* **05** (2003) 037 [hep-lat/0206019].

[8] A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a euclidean spacetime lattice. I: A target theory with four supercharges*, *J. High Energy Phys.* **08** (2003) 024 [hep-lat/0302017];
Supersymmetry on a euclidean spacetime lattice. II: Target theories with eight supercharges, *J. High Energy Phys.* **12** (2003) 031 [hep-lat/0307012].

[9] J. Nishimura, S.-J. Rey and F. Sugino, *Supersymetry on the noncommutative lattice*, *J. High Energy Phys.* **02** (2003) 032 [hep-lat/0301025];
J. Giedt, E. Poppitz and M. Rozali, *Deconstruction, lattice supersymmetry, anomalies and branes*, *J. High Energy Phys.* **03** (2003) 035 [hep-th/0301048];
J. Giedt, *Non-positive fermion determinants in lattice supersymmetry*, *Nucl. Phys.* **B 668** (2003) 138 [hep-lat/0304006]; *The fermion determinant in (4,4) 2d lattice super-Yang-Mills*, *Nucl. Phys.* **B 674** (2003) 259 [hep-lat/0307024].

[10] W. Bietenholz, *Exact supersymmetry on the lattice*, *Mod. Phys. Lett.* **A 14** (1999) 51 [hep-lat/9807010];
K. Itoh, M. Kato, H. Sawanaka, H. So and N. Ukita, *Novel approach to super Yang-Mills theory on lattice: exact fermionic symmetry and 'Ichimatsu' pattern*, *J. High Energy Phys.* **02** (2003) 033 [hep-lat/0210049];
S. Catterall and S. Karamov, *A lattice study of the two-dimensional Wess Zumino model*, *Phys. Rev.* **D 68** (2003) 014503 [hep-lat/0305002].

[11] S. Catterall, *Lattice supersymmetry and topological field theory*, *J. High Energy Phys.* **05** (2003) 038 [hep-lat/0301028].

[12] E. Witten, *Topological quantum field theory*, *Commun. Math. Phys.* **117** (1988) 353; *Introduction to cohomological field theories*, *Int. J. Mod. Phys.* **A 6** (1991) 2775.

[13] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[14] R. Dijkgraaf and G.W. Moore, *Balanced topological field theories*, *Commun. Math. Phys.* **185** (1997) 411 [hep-th/9608169].

[15] M. Blau and G. Thompson, *Aspects of $N(T) \geq 2$ topological gauge theories and D-branes*, *Nucl. Phys.* **B 492** (1997) 545 [hep-th/9612143];
J.M.F. Labastida and C. Lozano, *Mathai-Quillen formulation of twisted $N = 4$ supersymmetric gauge theories in four dimensions*, *Nucl. Phys.* **B 502** (1997) 741 [hep-th/9702106].

[16] P. van Nieuwenhuizen and A. Waldron, *On euclidean spinors and Wick rotations*, *Phys. Lett.* **B 389** (1996) 29 [hep-th/9608174];
M. Blau and G. Thompson, *Euclidean SYM theories by time reduction and special holonomy manifolds*, *Phys. Lett.* **B 415** (1997) 242 [hep-th/9706225].

[17] R. Brooks, D. Montano and J. Sonnenschein, *Gauge fixing and renormalization in topological quantum field theory*, *Phys. Lett.* **B 214** (1988) 91.

[18] L. Baulieu, H. Kanno and I.M. Singer, *Special quantum field theories in eight and other dimensions*, *Commun. Math. Phys.* **194** (1998) 149 [hep-th/9704167].

[19] H.B. Nielsen and M. Ninomiya, *Absence of neutrinos on a lattice. 1. Proof by homotopy theory*, *Nucl. Phys.* **B 185** (1981) 20, *ibid.* **B195** (1982) 541;
Absence of neutrinos on a lattice. 2. Intuitive topological proof, *Nucl. Phys.* **B 193** (1981) 173.

[20] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev.* **D 25** (1982) 2649;
P. Hasenfratz, *Prospects for perfect actions*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 53 [hep-lat/9709110];
H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B 417** (1998) 141 [hep-lat/9707022]; *More about exactly massless quarks on the lattice*, *Phys. Lett.* **B 427** (1998) 353 [hep-lat/9801031];
P. Hernandez, K. Jansen and M. Lüscher, *Locality properties of Neuberger's lattice Dirac operator*, *Nucl. Phys.* **B 552** (1999) 363 [hep-lat/9808010].

[21] M. Alvarez and J.M.F. Labastida, *Topological matter in four-dimensions*, *Nucl. Phys.* **B 437** (1995) 356 [hep-th/9404115];
S. Hyun, J. Park and J.-S. Park, *Spin-c topological QCD*, *Nucl. Phys.* **B 453** (1995) 199 [hep-th/9503201];
A. Sako and T. Sasaki, *Euler number of instanton moduli space and Seiberg-Witten invariants*, *J. Math. Phys.* **42** (2001) 130 [hep-th/0005262].

JHEP01(2004)015